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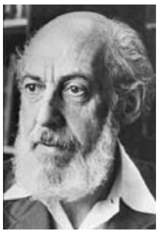
The Mystery of the Black Knight's Noetherian Ring

An investigation into the story-mathematics connection
with a small detour through chess country

A keynote address to the Fields Symposium on *Online mathematical investigation as a narrative experience,* Faculty of Education, University of Western Ontario, 11-13 June 2004



To the memory of Samuel Eilenberg,
co-inventor of Category Theory,
who thought I could become a mathematician.

Because something is happening here
But you are not absolutely certain what it is,
Are you, Monsieur Bourbaki?

Paramathematically adapted from
Bob Dylan's "Ballad of a thin man"

Less is more but it's not enough.
The Guerilla Girls

In a lecture given last year, entitled "Embedding mathematics in the soul: narrative as a tool in mathematics education",[1♣] I tried to summarize the experience and the arguments for using stories as a bridge facilitating students' access to mathematics. None of this was brilliantly original – as it need not be. We know now, after centuries of heated philosophical discussion, that people are much more than logical machines and so it is obvious that the royal road to a young person's brain – also to a not-so-young person's, for that matter -- is through the heart. And even the earliest homo sapiens knew that there's nothing like a good story to siege that most metaphorical of muscle groups. Math stories make people feel closer to math. Elementary, my dear Euclid.

My lecture was inspired by the *embarras de choix* of mathematical narratives that are becoming surprisingly (dangerously?) *à la mode* in recent years. Gone are the times when the Christmas math assignment had to be doing from number x to number y of the exercise book. Now the students can be given to read an exciting story about mathematics, real or fictional. Gone are the times when the prototype for the mathematician was a short-sighted, flat-footed, ridiculously absent-minded idiot savant. Now, it is Matt Damon and/or Russell Crowe – definite progress!

Still, I suspect there is much more to the math-story connection. The suspicion is largely motivated by my personal credo that all art, and especially narrative art, apart from any aesthetic-affective experience it provokes is basically a form of

♣ Numbers in superscript refer to endnotes, which are mostly bibliographical and/or free-associational. Non-numerical symbols refer to footnotes, which are as a rule parenthetical.

knowledge, an epistemological instrument[2]. And it is in this spirit that I approach the subject of the symposium, being no expert in – though quite an adept practitioner of – online investigation, but having some things to say about the relationship of mathematics to narrative that go beyond finding ways to make mathematics 'student-friendly' or huggable or whatever. And since this is principally a meeting of people separated by a common interest in mathematics education, I must also say this before I get into the subject proper: the reason I will not be talking much about education is because I believe that how we teach mathematics, as a culture, is shaped by how we do mathematics. And so, by addressing the issue of the paradigm shift – if you'll pardon the expression – that is taking place in mathematics in recent years and in which narrative can play a crucial part, I shall also be addressing, though mostly indirectly, the issue of how it can be taught.

♞♞♞

Being unfortunately an impulsive sort of person, and also being Greek, when a few years back I was invited to talk on the general topic of "mathematics and fiction" I immediately went in search of a general theory. In fact, I concocted a *Gedanken* experiment, to which I gave the somewhat grand title "Euclid's Poetics"[3], the basic idea of which is roughly this: since both a story and a proof share many of the characteristics of a quest (the first both in physical and conceptual space, the second purely in conceptual) it might be possible to establish structural equivalences (isomorphisms) between some of these totally different beings, thus setting up a sort of 'functor' – to use fancy language – between the categories of stories and theorems.

In a story, especially a classically constructed one like an epic or a mystery, the quest is the hero's attempt to reach his goal[♣]. (For Odysseus this would be returning to Ithaca, for Hercule Poirot finding, say, the murderer of Roger Ackroyd.) And a proof is of course also a quest, a logical journey from the premises to the final

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♣ Here I look at story types where the quest element is obvious, though the argument can be generalized to any type of story, which can be seen as a quest if we focus on the main characters' movements towards (or away from) their goals – all this is explained in more detail in "Euclid's Poetics".

statement. In “Euclid’s Poetics” I suggested that once the very different contexts are forgotten and we map the progress of either the hero or the mathematician as a path in some graph describing a space of possibilities, a proof and a story begin to look uncannily alike – as does an itinerary of a traveller in physical space, but moving in a higher dimensional, abstract space.

The thing to notice here is that both theorem proving and stories are about people in action to achieve a certain task – this is based on the assumption that mathematicians are people, you see. As a rule people have a limited bag of basic tricks to rely on to achieve their tasks, tricks that differ partially (but not totally) from domain to domain and can be augmented or refined by intelligence, imagination and/or courage.

No less an expert on the matter than George Polya gives the following list of tools (tricks) by which mathematicians solve problems:

- Guess and check
- Look for a pattern
- Draw a picture
- Solve a simpler problem
- Work backward
- Use a formula
- Make an orderly list
- Eliminate possibilities
- Use symmetry
- Consider special cases
- Use a model
- Use direct reasoning
- Be ingenious
- Solve an equation[4]

And anyone who, like the present author, has read enough mysteries can easily dash off a similar list describing a detective’s methods to solve a murder case:

- <u>Guess and check</u>
- <u>Look for a pattern</u> (consult criminal profiler)
- <u>Draw a picture</u> (from witnesses’ descriptions, if any)
- <u>Solve a simpler problem</u> (approach the investigation piecemeal)
- <u>Work backward</u> (see who profits by the crime)
- <u>Use a formula</u> (chemical, genetic, of whatever technology is pertinent)
- <u>Make an orderly list</u> (of suspects)

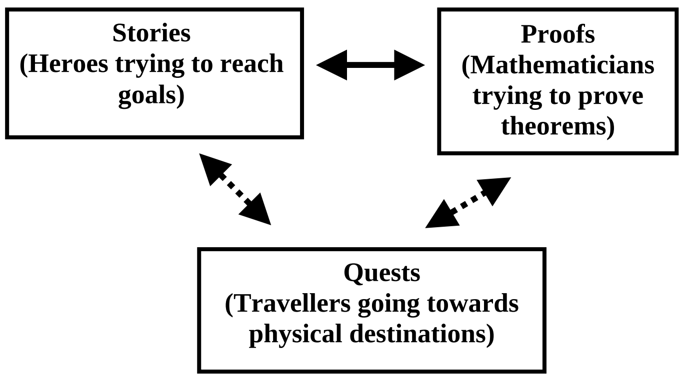
- <u>Eliminate possibilities</u> (through checking alibis, etc.)
- Search for material evidence
- Analyze it well
- Question witnesses
- <u>Use direct reasoning</u>
- <u>Be ingenious</u>

And possibly some more. As you can see, some items (underlined) appear in both lists, some not.

So, if we name the various stops in the two different journeys of discovery, mathematical and investigative, *a, b, c…* and *a', b', c'…* respectively, the stops on the first journey being propositions and those in the second really something of the same (i.e. discoveries of pertinent facts, elimination of suspects, etc.) a graph can be drawn with nodes *a, b, c…* in the first case and *a', b', c'…* in the second. The arrows connecting the nodes describe in each case the trick or syllogism or 'adventure' that the mathematician or the detective employs to move from one to the other. In both cases, the solution of the problem is a path on the graph from some initial point *a* or *a'* to some final destination *z* or *z'*. [5]

We see now that both paths are in some sense *stories*. For the detective this is obvious – at least since Edgar Allan Poe invented the genre. For the mathematician it becomes so if you forget proof as such for a moment and think of *narrating the mathematician's quest to solve the problem.* Obviously this story, as it becomes more and more detailed, requires mathematical background to understand – but that is alright, all stories have a context. At certain points, in a 'high resolution' narrative rendering of reality, the story may become *totally* mathematical, i.e. reducible to a formal language of the type "Let E be a Noetherian ring and *f* a homomorphism *f:* E → Ê, where Ê …", etc. But the thing to notice here is that this will happen *often but not always*, i.e. not all parts of the story can be reducible to mathematese. (More on this important point later.)

For clarity, I put the argument of "Euclid's Poetics" in the simple diagram below, saying that we arrive at the solid arrow (functor) on top via the transitive property, combining the two dotted arrows on the lower part. And this roundabout way of connecting proof and story seems handier since the spatial analogy is easier to establish for both[6]:



"General abstract nonsense", you may say♣. Maybe. But now the plot thickens.

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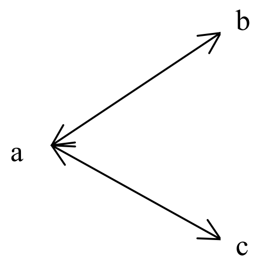
I gave the lecture on "Euclid's poetics" in April, 2001. And although it wasn't on April Fool's Day I have to admit that my functor connecting proofs and stories was offered somewhat tongue-in-cheek. Let us say that it was the work more of a writer than a mathematician, i.e. more of an inventor of interesting (hopefully) fictions than someone believing in rigorous proof. But this was done with the best intentions – and by this I mean in the belief that good interesting fictions can be helpful in the real world, if only to spur on a movement that may lead to something true. After all, scientists work with models and almost all models are untrue, at least in the sense of being over-simplifications of the realities they are supposed to portray, i.e. full of lies or omissions.

Now, speaking of functors, it is good to remember that the centripetal forces operating in any science, which tend to generalize from similarities (a task usually called induction) are always opposed by legion centrifugal forces, whose task is to identify differences. And in mathematics especially, this opposition has been at the root of most of the last century's epistemological squabbles. The major efforts to find a common unifying language for mathematics, mainly logicism ('all mathematics is reducible to logic'), the set-theoretical approach of Nicolas Bourbaki[7] ('all mathematics is reducible to set theory') as well as the attempt to reformulate the

♣ This is of course how Category Theory is affectionately (?) referred to by some mathematicians.

various branches in the Esperanto of Category Theory, have been supplanted by the sins of over-generalization: you see, what a language gains in generality, it loses – at least beyond a certain point – in revelatory power. The problem with *very* general languages (whether Category Theory or Business English) is that they speak economically only at a rather trivial level but rise to ungodly (and thus inhuman) levels of complexity (and pedantry) as soon as they must say something deeper♣. And I make these rather self-evident statements to underline the fact that my main fear in this investigation of affinities between the structure of narrative and mathematical proof is one of triviality. Indeed, I am heeding the words of Michael Atiyah: "The most useful piece of advice I would give to a mathematics student is always to suspect an impressive sounding theorem if it does not have a special case which is *both* simple *and* non-trivial."

And speaking of special cases, let's look at one of proof-story analogy. Take for example the simple path described by St. Basil's dog (see endnote 6) operating in a world where the Principle of the Excluded Middle holds: the graph below describes, in the generic sense, the structure that underlies all possible proofs employing the *reductio ad absurdum*.



Also, it describes all possible detective stories where there are only two suspects to begin with, *b* and *c*, and once *b* is eliminated (through establishing a watertight alibi, say), then *c* is certainly the culprit; also, all love stories where one's true love is discovered after going through all the ravages of being with one's not-so-true love, etc. All of the above are just the path *abac* on the graph.

Having warned myself against saying trivialities – a capital offence in mathematical circles –, I turn now to one of the main areas of focus of this

♣ See the *Principia Mathematica* and the 272 (or whatever) pages it took to prove "1+1=2". For the proof of the infinity of primes it would probably need Borges' Library of Babel.

symposium to address "the nature of mathematical experience and what makes for a good mathematics story". Now I may shock you by saying that I think that, from the point of view of good storytelling, "what makes a good mathematics story" is obvious: simply, what makes for a good mathematics story is what makes for a good story of *any kind*, i.e. interesting characters with goals that the reader can identify with, and a path from beginning to end that is zigzagging enough to keep the ride interesting – all this obviously in a context that is in some way mathematical♣.

But the really interesting question is not how mathematics can help create good stories (this being trivial) as *how storytelling can help create good mathematics* – and I hope I shall convince you that this is not so preposterous a task as it sounds at first. And, not to forget our reason for being here, let me say here that I think that any non-trivial answer to that question, however incomplete, will point to ways of applying it to mathematics education which go beyond the "turning-dislike-into-interest" function of mathematical stories.

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Before attempting to answer the question I just posed I shall make a detour, also incidentally throwing light on another element of my title: we have already more than once mentioned "mystery" and we even managed to get a "Noetherian ring" into our discussion. (Please remember, the storyteller's mentality drives me towards the concrete and exotic sounding.) Now it is time to introduce the "Black Knight".

I shall refer of course to the game of chess:

In his famous ('infamous'?) *Mathematician's Apology*, G.H. Hardy states that "a chess problem is genuine mathematics, but in some way it is 'trivial' mathematics." Now, it is interesting that in this credo of "pure mathematics" a chess

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♣ The only trick here, of course, is to make a mathematical goal appealing enough for identification. Rather than expand on this I shall state the title of Simon Singh's very popular book about the proof of Fermat's Last Theorem: *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem.* If 'enigma', 'epic quest' and 'world's greatest problem' is a combination good enough to attract us to Indiana Jones or James Bond, the publishers must have thought, then it's good enough for Pierre de Fermat – and so it proved to be.

problem♣ is considered to be mathematics, but not the colossally more complex, and more interesting, analysis of the game of chess itself. Of course, what is characteristic about "chess problems" as opposed to chess itself is their absolutely deterministic character, i.e. the fact that, once found, the solution points to an inescapable situation, a cul-de-sac for the losing side. Obviously, it is this which attracted Hardy, for in this determinism lies what a mathematician usually calls "elegance"[8].

The reason for my detour into the realm of Caissa♥ is that I believe chess is a good platform from which to address the questions of the paradigm shift that is taking place in mathematics in recent decades, from the Platonic-Hardyesque-Bourbakist-EAFist♣ to a messier, more open, less absolutist epistemology. And, what's more, it is a platform from which we can see to greater depth into the math-story link.

There were three reasons that started me thinking along the chess line:

a) The fact that this was one of Alan Turing's first goals when founding computer science: to create a machine that could beat a human player in chess. As everyone knows, this has been realized. And although it doesn't appear likely that a chess program *thinks like* a grandmaster, it can certainly beat one. (Think of the analogy with airplanes: the model for heavier-than-air-flight was the bird, and although airplanes do not flap their wings and could not run for a second on worms and seeds, they are certainly faster than any of their avian prototypes.)

b) My experience of trying to teach chess to my five-year-old son, where I found that the 'story' background of the game (the two warring nations metaphor) was rather useful as an introduction – *but not beyond a certain point.*

c) Reading the fiery Doron Zeilberger's 57th Opinion and his attack on arch-EAFist G.H.Hardy's snubbing of chess as mathematics[9].

♣ The term "chess problems", as used by Hardy, refers to the rather simple combinatorially, and thus of course 'trivial' in the mathematical sense, composed (not real-game) situations where one side can mate (only in the chess sense, of course) the other in a prescribed number of moves, usually no more than three. Of course strong chess players find these trivial as well!

♥ A dryad, the – of non-mythological origin – mythological goddess of chess.

♣ EAFist = adhering to the EAF (Esoteric Abstract Formalist) model; see my "Embedding mathematics in the soul", where this particular mentality is concretely attributed to its three Greek creators, Pythagoras, Plato and Euclid.

As chess-language is infested by military metaphors, I started my investigation by trying to translate chess games into stories. My first attempt started like this:

> CHESS STORY #1: *The White Queen sent forth to the field her trusted bodyguard, an excellent scout. The Black King, being a cautious man, sent out a horseman, to inspect his moves from a distance. Fearing some trickery, the White King had one of his Knight's soldiers cover the way for one of his fighting Bishops to approach the field from the side…* [10]

Now, this is very pedantic as storytelling and will become worse so, and extremely boring, if the description continues well into the game. And you can imagine that if one would want to make this style totally, unambiguously accurate – which might be altogether impossible – it would become so at the cost of greater length and even worse pedantry. Like that trite piece of popular wisdom comparing women and translations, our story version above will lose in beauty what it gains in accuracy. And even in its most meticulous (and pedantic) it will be a highly uneconomical description of a chess game, certainly much less so than the modern 'algebraic' description of the above as: 1. d4 Nf6 2. g3 d5 3. Bg2 e6, etc. And of course we all know that notation is of great importance in the development of thought – after all it was partly notation that did not allow ancient Greek, and then Renaissance, mathematics to progress further than it did.[11]

But as anyone at all familiar with chess knows there are two levels of methods of thinking, strategic and tactical, the second often reduced to its simplest form of calculating or "counting" (moves). This is the thinking of the type "if I take my Bishop to a3 he may take it with his Rook", i.e. what the *hoi polloi* equate with chess thinking. And, not surprisingly, this is exactly the kind of thinking that computers are excellent at, as they can calculate much faster, much more accurately and to much greater depth than any human being – and without ever blundering!

Here the first similarity with mathematics surfaces: obviously, a big part of mathematical thinking is also purely combinatorial-calculational. And – surprise, surprise! – this also happens to be just the kind of thinking that laypersons *equate* with mathematical thinking. (This is the view by which a mathematician is someone doing complicated sums or solving funny-looking equations on the blackboard.) But we know better: that this is only a *part* of mathematics and one that many good mathematicians dislike – in fact, it often goes against the requirements of elegance and the certain *je-ne-sais-quoi* at the heart of the classical EAF aesthetic.

Back to chess. When a player is thinking strategically, he or she is using a very different language from the "if I go there, he goes there" variety. A rather old-fashioned reductionist mentality – which, by the way, would be quite fine were it not for complexity and the truth in that old adage about "quantitative change becoming qualitative at some point" – might insist that any talk of "strategic" is but a mask for our ignorance and our inability to raise the tactical-computational to the absolute level it is entitled to (i.e. being able to calculate *everything*). But for human beings at their present stage of cognitive development, strategic-type thinking is not only useful but necessary – though alas not sufficient for all quests.

Now, a strategic-level rendering-into-story of a game of chess might go like:

CHESS STORY #2: *The White King marched his best soldiers to the center of the battlefield, but the Black King developed his flanks, preparing against an enemy attack by strong defensive measures. Clearly, the Black King was more intent to protect his kingdom, than to win. Yet, in the end his cautiousness and good sense led to victory. For when the White King made a risky attack, trying to break through to the Black King's camp with a sacrifice of his finest cavalry, the Black King immediately turned the situation to his advantage, crushing his opponent step by methodical step.*

Although the style of Chess Story #2 is not exactly Flaubert either, it certainly goes down better, as story, than Chess Story #1. And also, it tells us things about the game which – though certainly not sufficient to reconstruct it in every detail – are meaningful and significant to a chess player.

If we move one level up, from Kings and strategy to the human players' psychology and method, we get something like:

CHESS STORY #3: *When Vladimir Kramnik started to prepare for his championship match against Garry Kasparov, he was aware that he would be fighting a stronger opponent, quite possibly the strongest player who ever existed and, what's more, the one with the most profound knowledge of "opening theory" (the variations in the early phases of the game.) So Kramnik, being a very methodical and down-to-earth person, set out to* *prepare in a way that would neutralize Kasparov's "serve" with white, early in the game, in order to transfer the battle to the middlegame, where background research is less useful. He searched with his seconds and analysts in the annals of chess history, which Kasparov knew so well, and found that an old stratagem favored by ex-world champion Emanuel Lasker*[♣]*, an outmoded – and thus probably not very well studied by the otherwise very-up-to-date Kasparov – variation of the 'Ruy Lopez' opening, called the 'Berlin Defence' had a lot of hidden potential. So, he studied this in great depth, found many subtle variations, and used it every time he could when playing black. Kasparov was really totally surprised by the employment of an old and long-thought-passive defense which he did not know well enough – nor did he have enough time to study it during the match – and Kramnik managed to get four valuable draws in his games with black, i.e. in the games where Kasparov had the advantage of the first move and could aggressively go for a win. This was crucial in his final victory in the match...*[♥]

Now, this type of narrative thinking (for it *is* narrative and it *is* thinking) is extremely useful, mainly for two reasons: a) as chess is played by real people, human concerns, psychological, sociological or epistemological, play as important a part as the strategic and the tactical, especially in high level encounters; and, b) operating as it does on the level of metaphorical interpretation[♦], it is fundamental in building and maintaining the player's functioning at the symbolic-affective level which is so necessary for a winning mentality.

The gist of what we see by our three chess stories is:

- At the very detailed, tactical (and combinatorial-calculational) level, narrative rendering is all but useless, or worse: confusing and irrelevant.
- At the level of strategy it can offer some useful insights.

[♣] Incidentally, also a great mathematician. In 1905 Lasker introduced the notion of primary ideals and proved his Primary Decomposition Theorem, also known as the Lasker Decomposition Theorem, for ideals of polynomial rings.

[♥] It is extremely interesting in our context that the Berlin Defence is not easily amenable to analysis by existing computer programs – so it couldn't be taken apart by Kasparov's staff officers – as it's a more strategic variation, with no pronounced combinatorial features.

[♦] I mean this in the sense of it being full of words/concepts derived from the war metaphor: 'fighting', 'opponent', 'tactical', 'stratagem', 'defence', 'surprise', etc.

- At an even higher level (psychology, grand strategy, etc.) the story level is indispensable, both as a cognitive tool and at the level of motivation, whether positive or negative.

As it is impossible for a player to advance in the game without developing as much as possible his tactical and combinatorial-calculational skills, so it is a serious drawback for him or her who wishes to reach the highest levels to be ignorant of the history of the game and of the life stories of the great players. For chess, to a very serious player, does not begin and end on the board. It is a part of life and extra-chess factors come into it very significantly. As at the strategic level narrative is useful and at the highest level it prevails, storytelling[♣] is a necessary part of chess thinking and knowledge. Its importance can be ascertained by glancing through the contents of leading players in their own collections of their best games, where all three levels of analysis are employed. The relative importance of each level of analysis differs for each player, and establishes his/her personal style[♦]. Seen from a higher perspective, the evolution of styles describes the evolution of chess thinking, in its general form.

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I shall return to our chess stories and what they can tell us about storytelling helping us in doing good mathematics. But first I must face the question that begs to be asked: “what, in Archimedes name, *is* good mathematics?”

Well, as an outsider would certainly *not* expect – what with mathematics being ‘the Queen of the Sciences’, the domain of ‘absolute and certain knowledge’ and so on –, professional mathematicians have very differing views on this and their answers would form a highly inconsistent body of statements. But what is really interesting is that most, if not all, would agree that: a) the distinction between good and bad mathematics is highly meaningful and, b) in a way strangely smacking of Gödel’s theorems, the issue cannot be disputed (let alone settled) *within* any particular

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[♣] By the way, you can substitute ‘narrative discourse’ for ‘storytelling’, if you find the latter too light.

[♦] A small example: Kasparov does many more calculations than Karpov.

mathematical theory♥. And what's more – this *unlike* Gödel's theorems – it is extremely difficult to imagine that it can be resolved even at the level of a higher, more inclusive formal theory.

One does not need Gödel's genius to prove the endomathematical insolubility of the problem: after all, no mathematical theory contains a language adequate enough (actually: *vague* enough) in which even to state it. In fact, if the EAF model extended its strict demands from mathematics to thinking or talking *about* mathematics, a member of the totally *purus* species of mathematician, faced with the problem of 'goodness' would have to reply, like a good *entre deux guerres* logical positivist♥: "This is a pseudo-problem, a statement which is not just unprovable – and thus already unacceptable by strictly endomathematical EAF criteria – but unstatable. Quite literally, it is *non*-sense!" Yet, the intriguing thing is that mathematicians as a rule do not give this reply[12].

Hardy goes on and on about mathematical 'beauty' in his *Apology*, a quality he also ascribes to chess problems. But chess problems are "trivial", he says, because good mathematics has to be "*both beautiful and important*". Now, as Hardy couldn't give tuppence for anything as gross as *applications*, by "important" he obviously means "important for mathematics". And though one might try to define this criterion formally – oh, something having to do with generality, measured as the number of deeper results a theorem could lead to –, any attempt at definition would most probably run very soon into a vicious circle: 'It is important because… it is important…' Good grief!

The story of the impasse that mathematics reached in its navel-gazing "foundational crisis" period exercise has been frequently told: David Hilbert's dream of reconstructing a totally reference-free mathematics, consistent and complete, out of the most basic material, run up against the problems discovered by Kurt Gödel. What has not been given sufficient notice though is the lack of progress in the equally important need for mathematics to talk about itself *informally yet intelligently*. Unlike the foundational tragedy, this other failure has never even been acknowledged as a

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♥ It cannot be *formally* decided within the theory, i.e. the language of the theory itself does not suffice to discuss it.

♥ That is, by misinterpreting the concluding statement of Wittgenstein's *Tractatus*: 'What we cannot speak about, we must pass over in silence.'

serious one – in fact, I don't think that the problem has even been really noticed! – and this I guess mostly because of the blinders set in place by epistemological purism. Beliefs die hard, and despite any protestations to the contrary, almost all mathematicians tend to consider the EAF model as *the* epistemological tool to deal with any level of reality, however messy.[♣]

Hilbert's project of a mathematics so well-founded and rigorous that it can be totally derivable by a machine lacking the human frailties of imagination, intuition, etc.,[♥] was – like the tower of Babel – never completed. But still, untouched by Gödel is the fact that to whichever degree of combinatorial depth the Hilbertian ideal algorithm might go, the theorems it would produce would be *correct*. In that sense, they would definitely be mathematics – but *good* mathematics?

The Hilbert machine brings to mind the scenario of a hundred monkeys typing away: given sufficient time, they would produce all of Shakespeare's sonnets. Sure. Of course in olden times the problem would be you'd need a thousand humans and a few million years to extract the sonnets from the mess, though for a computer having the originals in its memory this would be a trivial task – given time. But if the task in question for the monkeys was not to produce all of Shakespeare's sonnets but some *good new sonnets* as well – and this is certainly no more daunting, statistically speaking – and for a computer to recognize them… Well, how could a computer do *that*? A sophisticated language recognition program could take the selection down to 14-line segments that are grammatically and syntactically ok, use acceptable words (though Shakespeare didn't always do that) and satisfy the metric and rhyming requirements of the form. But interestingly, setting up a program that would pick out the *good* poems would be probably as complex (or rather: equally impossible) as finding one to locate the *good* theorems out of the combinatorial diarrhea produced by an infinitely ticking away Hilbert machine.

Of course, hardly anyone believes anymore in the total logical determinism dreamed of by Laplace[13]. We now know that at every level of reality there are at least

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♣ This would explain the phenomenon that many great mathematicians are notoriously naïve when they theorize on matters exomathematical: they are unconsciously applying the EAF model to realities too complex to sustain it.

♥ His fellow superstar's Henri Poincaré's poignant joke was that this would be like a machine into which the pigs enter from one side and the sausages come out from the other.

some phenomena that are irreducible to the laws of a lower level. Chemistry is not totally reducible to physics and biology is not totally reducible to chemistry – though there are strong overlaps. Larger-scale thinking is as a rule required by the nature of a larger-scale reality. Of course, a mathematician can still be forgiven to think of mathematics as totally deterministic: it is after all the privilege of mathematics to deny reality, via abstraction, and thus to contain its own criteria of truth. What a mathematician does, after all, is create rules for one-person games that he or she then proceeds to play. The fact that some of these games are inspired by reality and some of their results can be applied to it, functioning like they do as models, does not – for many mathematicians at least – affect mathematics at all[14]. But all absolutist arguments have to be viewed with much more skepticism in the light of what we are learning about complexity.

And I now make the connection with chess: if our need for strategy (i.e. larger scale, non-formal thinking) is just a symptom of our being mental dwarfs complexity- and speed-wise, then all chess will be eventually reduced to calculating every path on a decision tree, and thus one day a computer will be able to play only perfect games. But complex as chess may be (the total number of possible chess games has been estimated to be about 10^{40} times more than the atoms of the universe), it is infinitely simpler than the body of all possible mathematical propositions – and this without taking into account mathematical fields or sub-fields that have not yet been developed. So: if strategic thinking is necessary in the two-person game of chess, why should it not be in the enormously more complex one-person games of mathematicians? After all, it is not the second player that adds the uncertainty making necessary the abandonment of solely deterministic thinking, but the ratio of the known to the unknown. And this is immense in mathematics, regardless of whether we are talking of the not-yet-known-but-in-principle-knowable propositions, or the in-principle-unknowable, the legacy bequeathed to us by Gödel?

♞♞♞

It is obvious that concepts like ‘beauty’, ‘goodness’, even ‘importance’ cannot be meaningfully discussed strictly within the limits of a formal mathematical theory. But what of ‘truth’ itself? At one level, that of ‘truth values’ of propositions, this is obviously within – indeed at the heart – of any formal theory. But I am thinking of Albert Einstein's comment to a mathematician: "My job is much more difficult than yours. What you say has to be right – what I say also has to be true." Could anything of the sort, i.e. a distinction of 'rightness' (for which a positive truth value would be enough) and 'trueness' be applicable to mathematics itself? ♣

You see, the problem with a great – if not the greatest – percentage of theorems coming out of a Hilbert-type 'machine' would not be that they are wrong but that they are *irrelevant* – as no doubt is some of the mathematics being produced today.[15] A decade ago it was calculated that approximately 200,000 new theorems were published every year. And although I'm sure that almost all of them are *right,* having been peer-reviewed, it is hard to believe that a very high percentage of them is a significant addition to knowledge – and a definition of truth must somehow allow it to connect with knowledge, or else of what inherent value is it?

Am I playing with the definition of words? I don't think so. If concepts like 'importance' (still using Hardy's word), 'goodness', 'beauty', 'relevance', 'interest' etc., are not within reach of a formal theory, then we should allow for the highest of philosophical values, truth, to be also less-than-perfectly formalizable, even when applied to mathematics.

I am treading on dangerous ground here, and I must be careful. I want to make it absolutely clear that I am not siding here with the promoters of such inanities as 'feminist algebra', or 'Aryan cohomology theory' or whatnot. Even before Andrew Wiles and his proof, a person who seriously argued that whether $a^n + b^n = c^n$ has integer solutions for $n>2$ is a matter of taste, or race, or socioeconomic factors, would be rightly considered a crank.

Clearly, any theorem established by peer-accepted proof within a given axiomatic system with well-defined rules of inference is 'right'. But no amount of proving can convince a mathematician to believe that Fermat's Last Theorem is as

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♣ Assume for example that I now invent and propose arbitrary new definitions and an axiom system which is consistent, etc. This is my mathematical theory, my one-person game. Any statements I produce **in it the rigorous way** are 'right'. But can they be awarded the accolade of 'truth'?

important as all the hype tells us it is.♣ Yet, there is no chance in a zillion that Wiles would have become a front-page celebrity, or the story of his effort a bestseller, if it was simply said of him that he proved the Shimura-Taniyama Conjecture, which is in fact exactly what he did. But to us writers this comes as no surprise: we have learned the hard way that human society feeds much more on the mythological (which is of course narrative) than the rational. And what myth-eater worth the name would want the Shimura-Taniyama conjecture, given also a choice of “the world’s greatest mathematical problem” as FLT became to the media, for a while anyway?

Richard Feynman has said that “mathematicians can prove only trivial theorems, because every theorem that is proved is trivial”. And although this is certainly partly a joke on mathematicians’ use of certain concepts, there is an element of truth in it: known (i.e. proven) mathematics is in a certain sense “obvious”. But that is where formal theories stop. If we move into the sphere of the unknown, the EAF criteria wither. For example, there are many excellent mathematicians who *believe* that the Riemann Hypothesis is true but there are also some, equally excellent, who *believe* it probably is only partly true. And what of all other results of which there is no proof, or even further, those of which even the basic premises have not been defined? What is the value of that part of mathematics for which the only formally acceptable criteria cannot be established? And when we say this, we must remember that it is almost certain that infinitely more mathematics is unknown than known. Are we to exile it from our discourse, must we simply relegate it – if we can describe it well enough – to the “to be possibly proven” list?

Von Neumann’s warning is certainly to be taken seriously: “As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from ‘reality’, it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*...” Yet, this is but one central aspect of the problem. We must not become philistines and say that a direct relationship to exomathematical applications is the sole criterion of importance for mathematics. It is basic; but by no means the only one.

♣ In fact, Wiles’ real contribution to mathematics was that he proved a *much more general* result, of which FLT was a corollary, as had been previously been proven by Ken Ribet working on a conjecture of Gerhard Frey.

So, if not at applicability, where else should we look for the meaning of mathematics, the locus of the valuation of the criteria of 'good' and 'important'? We mentioned earlier that if we try to solve the problem with purely endomathematical criteria it most probably tends to become circuitous. And more than a hundred years of philosophy of mathematics have not really contributed to answering these questions. We do not yet have a score on which of the traditional camps got more points right – though all the talk about formalism vs. logicism vs. intuitionism, etc., has undeniably spurred-on progress in some mathematical fields[♣]. And today Platonists, formalists, logicists, intuitionists and constructivists alike, all have to take at least partly into account the not-really-philosophical Darwinian-phylogenetic slant that has recently become very (too?) popular when talking about the ontology of human knowledge of any kind. Indeed, anyone but an ultra-fanatical zealot of the old schools would have to admit that if we take the Platonist and the constructivist viewpoints as two extremes, cognitive science teaches us there is at least an element of truth in both. In the Platonist extreme: it is difficult to see how the natural numbers or π can somehow *not* be inherent in the nature of the cosmos ("made by the good God", Leopold Kronecker would say). And in the other end, the constructivist: it is impossible not to admit that at least *some* mathematics, oh for example Topos Theory, or Wiles' way to Fermat, or for that matter FORTRAN – which is not more of an artifice than Topos Theory – is at least partly (heavily) molded by the freedom of human creativity.

But although some mathematicians take an interest – by no means all do – in the philosophy of mathematics, formal or not, and many enjoy reading either scholarly or general accounts of the history of mathematics and the biographies of mathematicians, hardly any important mathematicians take these matters really seriously – I mean seriously enough to affect their work – at least while they are in the peak of their careers[16]. To most mathematicians, any discourse about mathematics outside the EAF model is not doing mathematics but talking shop.

♣ I think that the most impressive case of this is Kurt Gödel's statement that he could not have conceived of his theorems, in the first place, if he was not a committed Platonist in his mathematical philosophy, i.e. if he did not accept independent existence of mathematical truth.

♞♞♞

In fact, the human activity we call mathematics is traditionally considered to be practiced in two ways:

a. The endomathematical of 'mathematics' per se (sometimes also identified as 'pure') in the sense of formal theories. Here abstraction is of the essence, and truth is equivalent to rightness, i.e. defined by what is rigorously derivable from a particular set of axioms. 'Mathematics' defines the only acceptable epistemological criteria for mathematical truth. It is, officially at least, self-contained and self-motivated, sometimes to the extent of solipsism. (But solipsism is not-formally definable.)

b. The exomathematical, operating inside or at the limits of the sciences (sometimes also called 'applied mathematics'). Up to the middle of the 20th century, practically the only non-trivial exomathematical interaction of 'mathematics' was with physics and statistics. But now, in addition to these, computer science, biology, meteorology, economics and other fields have developed bridges to 'mathematics'. And apart from using strong mathematical methods themselves, they contribute to the creation of new endomathematical ideas.

Of course, 'mathematics' (as defined in a. above) forms, and most probably will continue to form, the core of the field, the sea into which the various rivers of applications (b.) flow to become 'mathematics' by succumbing to the rigid, inflexible epistemology, abstract, formal and rigorous, that is its distinguishing characteristic. But in the past few decades, some people – modesty prevents me from saying 'some serious people' – feel that the omnipotence of the EAF view should not go unchallenged. More specifically:

c. A new field called 'experimental mathematics' has come into existence. (Not everyone uses the term in the same way.) And although only few of its practitioners go as far as proposing for it a Popperian, natural science-type epistemology♣, the colossal calculating power of computers is supplanting the autocracy of the EAF model, at least in the sense of upsetting the traditional criteria of proof and showing that powerful, deep results do not necessarily have to be elegant[17].

And it is in this same spirit, though in a very different sense, that I recently proposed in "Embedding mathematics in the soul" not so much adding to the above three senses as acknowledging the existence – and, what's more, the *importance* – of a fourth, valid way of dabbling in the wider of mathematical truth:

d. *Paramathematics*[18], i.e. the multidisciplinary (and – correctly – undisciplined) field, lying somewhere in the overlaps of the history of mathematics, mathematical biography, the cognitive psychology of mathematics, the philosophy of mathematics (mostly in its non-formal forms♥), the history of ideas, relevant branches of the history and philosophy of other sciences, and so on, whose aim is to discuss the development of mathematics in a non-formal context, mostly in the narrative mode.

The mode of thinking in this case is the narrative because paramathematics does not aim at reaching abstract, irrefutable proofs, as EAF 'mathematics' does, nor at studying mathematical truth as a natural phenomenon (as does a lot of 'experimental mathematics'). Unlike the philosophy of mathematics which operates with its own a priori principles – and thus has little valid interaction with 'mathematics' – or fields like the cognitive psychology of mathematical knowledge, which follow the epistemological model of the natural sciences, paramathematics operates within the criteria of the 'narrative mode of knowing' as defined by Bruner (endnote 2). It accepts human nature as the locus both of the creation and the reception of mathematical knowledge, thus deals both with mathematics and human beings and values, a domain where there cannot be absolute or final knowledge. The field par excellence for paramathematics is the stories of the questions, the methods,

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♣ Which would allow, for example, the Riemann Hypothesis to be accepted as in some way 'true' until a counterexample is found.

♥ We must not forget that a lot of the so-called philosophy of mathematics, as well as all of metamathematics (the field concerned with the study of mathematical theories) are really 'mathematics' in the sense of being founded and operating totally on the EAF model.

the problems and the solutions defining mathematics. In fact, this is already the most central concern of paramathematics, as the budding, half-formed, slowly-finding-its-form literature of the field attests: thinking non-trivially about problems through their history. And anyone who believes in storytelling as a mode of knowing has probably learned the lesson, that to create a good story about something is definitely a way to think non-trivially about it.

What is crucial to realize here – and like a mathematical truth, almost obvious once realized – is that in some sense *the story of the solution of a problem is more important than the solution of the problem itself*. And this is so – to keep things simple – already because of the basic fact that *it contains it*. In other words, the retelling in the rich context which includes biographical, historical, philosophical and other factors, of the story of the attempts at, or the actual solution of a problem – both story and proof, remember, are the putting into a certain order propositions that get you naturally from beginning to end – contributes to the understanding of the solution-as-discovered (which of course is not the same as the solution-as-published) and in this sense contributes valuably to the exploration of the general domain of mathematical truth.

Car-making is not just the cars, and mathematics is not just the theorems. Mathematics is a complex human activity in which the EAF model and its custom-built rigorous epistemology applies only at the top end – or some end, anyway. We have to realize that by studying the problems and the quests for their solutions we are not studying the history of mathematics but actually doing mathematics, though not in the EAF mode. We are dealing with mathematics and achieving a synthesis which can be both interesting and productive in itself and also, occasionally, lead to new and important 'mathematics'[19].

And to those who think that by all this I am trivializing the mathematical method, or distorting its basic premise by reducing hardcore mathematical thinking to storytelling, I remind the three chess stories, the three levels, tactical-computational, strategic, cognitive/psychological/historical. As in chess, so in mathematics storytelling operates at three levels:

a. Although a syllogism may be structured like a story (and in this sense is a story) at the calculational and/or rigorously deductive level narrative is probably of little or no value as an exploratory tool.

b. At the higher, 'strategic' level of understanding, the structure of an argument becomes more important than the nitty-gritty of getting from *a* to *b*, and thus the narrative mode can be more pertinent.

c. At the highest, historical-biographical-epistemological level, storytelling is predominant. It is here, especially, that questions of value can enter and be thought about, narratively, as in no other form of mathematical activity. Let's not forget that mathematical history has not claimed up to this point such depth – it has not been written as belonging to the history of ideas –, exhausting itself mostly in mere biography or the chronicling of certain branches. After all, it was often practiced by retired mathematicians, conforming to Hardy's ridiculous belief that older mathematicians are "second-rate minds" (sic) and thus only fit for "exposition, criticism, appreciation" (see the first paragraph of the *Apology*.)

Thus, paramathematics operates differently at different levels of complexity. We must understand that it is alright to be non-rigorous when doing mathematics – *as long as you are not claiming that you are so being*. And, after all, you can only be rigorous at the formal (the lowest, in terms of complexity) level of knowledge.

♞♞♞

By way of conclusion, and turning full circle to touch base inside the domain of investigation of this symposium, I want to repeat that the application of the paramathematical mode of thinking to mathematics education is quite direct. By defining the learning of mathematics as not just the introduction to the EAF model, which is what we have essentially been doing for a couple of millennia, but *also* as the narrative study of its problems – there is no suggestion of replacement, only of addition to the EAF model –, we are enlarging the scope of the teaching of mathematics (which almost exclusively meant 'mathematics' up to now) by making it a field that is much closer to the concerns of young human beings. And this is because, unlike 'mathematics' itself, the narrative exploration of its quests is made of the same stuff as life-as-lived or, at least, of a stuff much more reminiscent of its complexities and vicissitudes – and no human being is, alas, too young to have a healthy dose of the complexities and vicissitudes of life – and thus is more important, in all senses of the word.

Acknowledgments

I wish to thank Professors Christos Papadimitriou and Doron Zeilberger for reading this paper and making many valuable comments. Also, chess International Grandmaster Nigel Short and International Master Ilias Kourkounakis for inspiring discussions and comments on my excursion through chess country.

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[1] Available online at http://www.apostolosdoxiadis.com/files/essays/embeddingmath.pdf

[2] Though I arrived at it through different byways, this view is wonderfully (and famously) put forth in Jerome Bruner's "Two modes of knowing" (In Jerome Bruner, *Actual minds, possible worlds*, Cambridge: Harvard University Press, 1986.) In this article, narrative is convincingly argued to be one of our two basic tools for understanding the world – the other of course being the more acknowledged classificatory-deductive method of science.

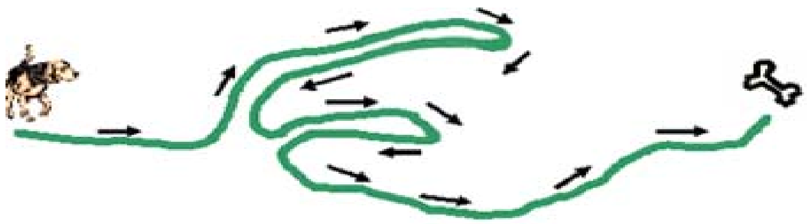
[3] Available online at http://www.apostolosdoxiadis.com/files/essays/euclidspoetics.pdf

[4] "George Polya" by A. Motter, at http://www.math.wichita.edu/history/men/polya.html

[5] We notice that for the mathematician proving theorem A, there are really *two* graphs (or, if you want, two paths on the same, more inclusive graph) the first being the course he or she followed to *originally* prove A and the second the – as a rule much more economical and rigorous – course that will be used in presenting the proof in writing, in accordance with the *lege artis*. In the case of the mystery this other path would be the sequence of the logical presentation of the evidence at the trial, obviously totally different from the messy, searching form in which it was discovered. In fact, in each case the first story will be the story of the discovery, while the second will be the story of the crime – an interesting distinction to keep in mind.

[6] The first impetus for these associations came from the unlikeliest source: a theological treatise called *Homilies on the Six Days of Creation* (4^{th} century) where the author, Saint Basil of Cappadocia, propounds the extraordinary idea that mathematicians invented the *reductio ad absurdum* by watching dogs search for food. More precisely, the dog in the diagram below, having smelled a bone in a certain direction will first explore one possible path to it, and if he doesn't find it will not automatically correct his trajectory – if he did that he would have taught us the method of *successive approximations* –, but will go back to the beginning, modify his premise – his course – and try again. (The dog in this

particular diagram has never heard of the Rule of the Excluded Middle, so he has to do it twice.)



This is really an amazingly pregnant observation, which – apart from probably being true in the "where does mathematics come from" sense --, links three concepts: a quest story, mathematical investigation, and a course in physical space.

[7] 'Nicolas Bourbaki' is the name collectively used by a group of French, mostly, important mathematicians who back in the thirties started the grand project of founding the whole of mathematics on a very refined, formal version of set theory – their opus magnum remains incomplete to this day. What I find amusing in the context of this discussion is that Nicola Bourbaki, i.e. the prime modern advocate of the EAF (Esoteric Abstract Formal) model of mathematics which was founded by the Greeks, mainly Pythagoras, Plato and Euclid, is also in a roundabout way of Greek origin: the name 'Bourbaki' originally entered France as that of General Charles Denis Bourbaki, a renowned 19th century military man who was the son of Colonel Vourvachis (Βούρβαχης) a fighter in the Greek war of independence. (Interestingly, Charles Denis Bourbaki was offered – and refused! – the then vacant Greek throne in the 1860's.)

8 Doron Zeilberger, in a personal communication, says that even in the initial position of a chess game, "the big problem 'Can White Win' is equivalent to 'Mate in <=200 moves', so it is a 'chess problem'." Though of course this is a thought-provoking observation, it does rather extend the accepted usage of the "chess problem". The theoretically longest-possible chess game has been calculated to last 5,899 moves. But of course, 99.99% of serious games are well into Zeilberger's 'under 200' moves limit.

[9] http://www.math.rutgers.edu/~zeilberg/Opinion57.html

[10] A chess player will see that this is a somewhat messy variation of the start of the Catalan Opening.

[11] As late as the 17th century, Pierre de Fermat stated his famous 'last theorem' as "cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere", a statement which would now be "$x^n + y^n = z^n$ has no non-zero integer solutions for n>2" – from 156 down to 43 bytes.

[12] I at least have never met a mathematician who did not passionately adhere, whether consciously or not, to both an ethics – if I may so call a discussion of 'good' and 'bad' – and an aesthetics of mathematics. In fact, every mathematician has read and I think most believe in the statement of G. H. Hardy "there is no place in the world for ugly mathematics".

[13] "An intellect which at any given moment knew all the forces that animate nature and the mutual positions of the beings that comprise it, if this intellect were vast enough to submit

its data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom. For such an intellect nothing would be uncertain; and the future just like the past would be present before his eyes." Pierre Simon de Laplace, *Philosophical Essay on Probabilities.*

[14] For a taste of the rich discourse on this, see G. H. Hardy's *A Mathematician's Apology* (Cambridge University Press, 1940); Eugene Wigner's famous article "The unreasonable effectiveness of mathematics in the natural sciences" (available at http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html; Paul Halmos' "Applied mathematics is bad mathematics" (1981) in *Mathematics tomorrow* (L.A. Steen, ed.), 9-20, Springer, New York; Doron Zeilberger's "People who believe that Applied Math is Bad Math are Bad Mathematicians" (http://www.math.rutgers.edu/~zeilberg/Opinion2.html)

[15] I am grateful to Professor Doron Zeilberger for his comment on this point: "(…I think that you) succumb to Hardyian elitism by saying that a significant amount 'of the math produced today is irrelevant', maybe the results are, but the *activity* of producing new theorems is the *message*, the fact that there exists a *culture*, and the whole is much more important than the sum of its parts. So it is nice that there exists a living language called math that is spoken (or rather written) at levels. So that's another analogy with story-telling: There is a place in the world for mediocre and even bad literature/fiction, if nothing else so that the good writers will stick out!"

[16] Field medallists Alain Connes and Timothy Gowers are among the few brilliant exceptions to this.

[17] The most characteristic example of this is the Appel-Haken proof of the Four Color Conjecture, where the solution was derived by reducing all possible maps to 1,936 cases whose 4-colorability was checked by computer.

[18] Paramathematics, from the Greek 'para-' meaning 'at the side of' as in parallel ('at the side of another') or paradox ('at the side of *doxa*, i.e. belief').

[19] When I define paramathematics as, mostly, the 'stories of problems', I do not mean that any account of the story of a problem is of intrinsic value to mathematics. A book such as Singh's on Fermat's Last Theorem, though a great read and a fascinating introduction for the non-mathematical public to a famous problem, does not in any way shed additional light to it. Yet other books – the list is not exhaustive – like Martin Davis's *The Universal Computer*, Peter Pesic's *Abel's Proof*, Amir R. Alexander's *Geometrical Landscapes* and Karl Sabbagh's *The Riemann Hypothesis*, contribute I believe, if not to the advancement then at least to the deepening of our knowledge of mathematics by telling the stories of problems in an interesting way and adding sophisticated 'para-' syllogisms to the formal development. It is worth noting that the author of the last of these books – incidentally republished two years after its original publication as *Dr. Riemann's Zeros* (!!!) – is a documentary producer who studied anthropology and employs in his interviews the approach of a field anthropologist trying to discover the *Weltanschauung* of mathematicians. So: paramathematics is not just for mathematicians, manqué or otherwise!

Apostolos Doxiadis

“The Mystery of the Black Knight’s Noetherian Ring”

A response to comments and questions

To begin with, I would like to thank everyone who contributed comments and questions, (“c & q”) in response to my paper. Reading them over a few times, I feel that by reacting I can make my thesis more complete and, hopefully, clearer – again, thank you for this opportunity.

I would like to begin by making two *General points*. Without referring to specific c & q’s in them, I believe that they address, in a rather pedantic idiom, a significant range of the issues raised. But since the c & q’s go into some detail and many raise specific points or objections, I would also like to answer those in *Responses to specific c & q’s*.

General point 1

The field that I label as ‘paramathematics’ can be more or less adequately described as “the study of mathematical history and/or biography from a particular point of view”, this point view being a combination of an epistemology and a rhetoric, mixed to varying degrees in each particular case. (Incidentally, I note here that though I ardently promote the need for the field’s existence, I have no very strong feelings concerning my particular choice of name for it – what’s in a name, after all?)

Thus, when the term ‘narrative’ or ‘story’ is used in a mathematical context in my paper, it means more often than not ‘stories of mathematical problems’. In other words, as far as its domain of enquiry is concerned, paramathematics is a full subset of the universe of mathematical history and biography. However, it is the particular point(s) of view that I am advocating that set this approach apart from most, but by no means all, existing historical and/or biographical work in mathematics.

I referred to an *epistemology* and a *rhetoric* as biasing the stance of a paramathematical narrator. The *epistemological* slant is an emphasis on ‘how we know things' that is informed by the classical concerns of philosophical epistemology, as well as the epistemology of science and mathematics, and the more modern insights and techniques of fields within the general area of the cognitive sciences. This can move from the 'abstract' end of the spectrum, where the question 'what is mathematical knowledge' predominates, to the totally human-centered, involving issues of how human beings learn, discover and/or know mathematics – and these both at the general level but also applied to specific problems or fields. On the other hand, the *rhetorical* concern is not so much influenced by the need to define, discover or create this kind of narrative, as to effectively communicate it. And this element of paramathematics is, obviously, more relevant to issues of teaching. So: at the one end, a paramathematical narrative with strong emphasis on epistemology will concern itself with how and why a particular mathematical 'story' developed the way it did; but all or part of the same story can be told – at the other end – with a rhetorical concern, i.e. an emphasis on it being better understood. Obviously, the two aims, epistemological and rhetorical, can often overlap.

The epistemologically-biased paramathematical narratives have to do with the practice of mathematics, with how mathematics is done. **The rhetorical cater more to what we may call, also, 'mathematics education'.** I quote from the article on Elena Nardi's work referred to by Bill Higginson in c &/or q's: "Accessibility is often the underlying intention when mathematical learning is reduced to an execution of cues and procedures. And yet, devoid of rationale for their use, these procedures often seem mystifying and alienating and are strongly resented by the students." If we take the word "context" as wider than – and containing -- "rationale", a rhetorical/paramathematical approach to the history of mathematics is a tool for making the field more meaningful to students, via the creation of context.

One of the central defining characteristics of storytelling is its *concreteness*, and though generic versions of stories do exist (i.e. 'boy meets girl, boy marries girl, girl abandons boy for other boy'), usually the more specific a story is – and this in more or less full contrast to the theorems of mathematics, where generality often goes hand in hand with importance –, the more effective it can be. Obviously, the need for injecting concreteness to mathematical education goes all the way back to the simple (but not psychologically trivial) case of teaching young human beings '2+3' through

‘John has two marbles and Mary gives him three more’ – perhaps this is something approaching a universal species-reflex against the abstraction and generalization inherent in the subject, that many (most?) students find such a barrier to the mathematical world. But though problems can be only be made more interesting to a certain extent by ‘concretizing’ them, whether in books or interactive contexts, this kind of injection of the real world into mathematics goes against the very essence of mathematical thinking which, beyond a certain level of complexity, is almost synonymous with abstraction.

Thus, although the ‘concretization’ approach to problems in a sense antagonizes mathematical thinking, a certain way of telling mathematical history and biography (precisely: *paramathematics*) can remain concrete at any level of complexity of the underlying mathematics, referring as it does to a human underlying reality – which has to be concrete, to have existed! The story of the field and its subfields, as well as of the people who created them and the issues involved are always concrete – and there is no limit to the sophistication this discourse can go to. From ‘one plus one is two’ to Noetherian rings, there is a story to tell, a complex story of discovery, that can admit as much rigor and sophistication as we want in its more concretely mathematical arguments.

General point 2

This brings me to the second point: that paramathematics in no way aims at replacing – Euclid forbid! – mathematics, either as a discipline or a form of teaching. I strongly believe it is a *necessary* complement to mathematics though; and, in fact, paramathematics is something mathematicians have been doing all the time on the side of their mathematics, though they have not spoken about it except very recently. Of course it is in no way *sufficient*, though.

So: what I am really suggesting in my paper is that the acceptance of the fact of a paramathematical intelligence operating on the side of the mathematical, in the mathematician, the teacher and the student, enriches the field and its understanding as it forms its natural *context*, which is of course by itself a “rationale for its use”. A rationale does not have to be of the type “arithmetic it is useful to keep account of your spending" or "calculus is necessary to build rockets/cars/computers…" A rationale for human action needs cerebral justification only when it is not supported emotionally -- one of the ego's prime defense mechanisms is called, after all, *rationalization*. (This is what is usually meant by a response "if you have to ask, the answer is no".) And only a storied universe can provide this richness of meaning, necessary to make things acceptable without specific rational explanation. Mathematicians operate, live, are motivated and enriched by this world or paramathematics – i.e. they live inside the ever-developing story of mathematics. How do we expect students and teachers to do mathematics being completely outside it?

Mathematics and paramathematics are different. This is an important truth. But it should not blind us to their areas of overlap. It is precisely for this reason that I speak of the 'three levels' of operation of a story in mathematics. (Of course there is a much richer stratification, the three levels marking significant positions on a curve.) And to further clarify this point, I rephrase it: what I am saying, in essence, is that only certain parts of the mathematical process can be fruitfully transferred into stories. At the lower level of complexity, the computational/formal level, this is practically nil. At the other end, the large scale/bird's eye view/higher/historical, it becomes extremely important. What happens in between depends on the particular slant of a narrative and its function. Sometimes a narrative recounting the progress of a mathematical field with minimal formalism, and even emphasis on non-mathematical criteria, is all important. Sometimes formalist rigor is required. But as extreme mathematical rigor usually enters a field *after* its creation – sometimes with detrimental consequences to the creators, as Andrew Wiles almost found out when the gap in his original proof of Fermat's Last Theorem was discovered – so mathematicians (and one would think: teachers and students) can talk about and even do mathematics (*sometimes!*) non-rigorously but significantly. And when there is no need for rigor, that is the time for the narrative thinking to creep in.

Obviously, the distinctions operating at the three levels concern the *part* of a paramathematical discussion directly concerned with the mathematics. I mean of course that a paramathematical 'quest story' will certainly contain aspects like the psychological, personal, social, general historical, etc. that – though they may be important to the creation of the mathematics – need not concern themselves in any way with formal arguments, being more about the creators than the creations.

I now move on to:

Responses to specific c & q's

I respond the c & q's in the order in which they were sent to me. Please excuse my omissions, or the rather carefree style, which is really the result of my time pressure – as a chess player might say. For reasons of completeness, I quote at least the part of a comment or question that intrigued me, or requires a direct response in blue. However, to do full justice to the comments and questions as they were posed, I must refer the reader to their full text.

Bill Higginson

..The problematic reality of contemporary learners of mathematics is not that of uber-rarification, but rather, the ultra-fatigue of T.I.R.E.D. This acronym arose out of a study carried out by the perceptive (Greek) researcher Elena Nardi and her colleagues at the University of East Anglia a few years ago: *A new ESRC-funded report shows that quiet disaffection is ever more evident in the secondary school classroom. In fact students are literally T.I.R.E.D. of maths according to a new profile which includes the characteristics Tedium, Isolation, Rote learning, Elitism and Depersonalisation.* **If this is the case, what are the implications for the creators of mathematics learning materials (perhaps online, possibly narrative)?**

I think this is one of the basic points addressed by my main thesis: that all of the five elements of TIRED can be – if not eliminated – then at least significantly counteracted by an emphasis on context-building paramathematical narratives, setting mathematics in historical, epistemological and human context. The implications for the creators of materials are: try and go back to story, as much as possible. First make them love it, then do it – the other way around does not necessarily work.

I'd like to hear Mr. Doxiadis' views on the issue of weaknesses and limitations of a narrative approach.

Obviously, the narrative approach is not a panacea for anything – especially not in the case of mathematics. It is a valuable tool to work side-by-side with more traditional tools. I hope I am allowed a comment on Galen Strawson's review of Jerome Bruner's *Making stories: Law, Literature, Life* that gave rise to this particular comment:

I had not read the review before Professor Higginson pointed it out in his comment, but I had read Bruner's book and thought very highly of it. Galen Strawson is an analytic philosopher and his view is definitely coloured by the prejudices of that school: a school that delights in branding as 'nonsense' anything that cannot be reduced to facts deducible from primary sense impressions and a formal calculus of elaboration thereof, religion, metaphysics, aesthetics, and ethics have been among their targets. And although he is undoubtedly right that a lot of the 'storying' craze is nothing but a (passing, certainly) fad in the social sciences, his objections cannot be generalized to the whole field of narrative inquiry. Of course, fanatical adherence to any rigid point of view – analytic philosophy included – definitely obscures more truth than it reveals. But to criticize a whole, fruitful and profound approach to certain aspects of human activity because of its excesses, is to throw out the baby with the bathwater.

…Researchers like Baron-Cohen at Cambridge in their work on autism are beginning to identify a close link in many cases between mathematics and what he [unfortunately in my view] calls "the extreme male brain"… Quite a lot of the avalanche of paramathematical [there's that word again] 'literature' (using the term broadly to incorporate film and drama) - including Mr. Doxiadis' very fine *Uncle Petros* – dance around the delicate issues of (shall we say) *eccentricity* and *intensity/obsession*.

I'd be very interested in hearing Mr. Doxiadis' views about this possible set of connections between and among abstraction, empathy (or the lack thereof), personality and narrative.

To me there does seem to be a significant connection between the *eccentricity* and *intensity/obsession* and the creation and understanding of mathematics. Of course, writers and/or directors often stress this aspect for its 'media value' (or whatever):

readers and viewers have always been attracted by a good dose of madness in works of biography or fiction. However, the relationship is deeper, going down to an argument that is, roughly: mathematics as an activity is really *systemizing* (by Baron-Cohen's terminology) taken to extremes. To systematize the world you need to simplify it – a process that in mathematics is almost synonymous with abstraction. But as abstraction is to a large degree a Procrustean operation (i.e. one of getting to the essence by discarding huge chunks of reality as 'irrelevant') it is best supported/carried out by people who have a psychological makeup that can relate more easily to a more fragmented – and thus less *balanced* – Weltanschauung. People are not logical machines, except very partially. To operate adequately as such – i.e. with this particular bias to the exclusion of others – it helps to have a personality structure that can go into a 'purely intellectual' mode more easily. As the *Curious incident of the dog in the nighttime* brilliantly demonstrated, this kind of approach, taken to extremes, is highly pathological. Of course, not all mathematicians suffer from autism, Asperger's Syndrome, paranoid - or obsessive/compulsive personality disorder. But the psychological processes that *in extremis* characterize these afflictions are at least partly essential to mathematics, a field that is abstract, formal, combinatorial, meticulous, occasionally extremely sensitive to the minutest error. It is these parts of our functioning, if distilled and applied, that create a lot – though not all of mathematics. And though we do not have to be overall pathological to be mathematicians, the part of us that does mathematics, if it became dominant in a personality, would definitely lead to pathology, to the total lack of that *negative capability* that John Keats defined as so necessary for a strong poetic relationship with the world – this cannot be excluded from any meaningful human life. This is a huge and largely unexplored subject, though. Ben-Ami Scharfstein's book *The Philosophers: Their Lives and the Nature of their Thought* (Basil Blackwell, 1980) contains some relevant material, especially relating to the people behind the 'foundational crisis' of mathematics, in the late 19th and early 20th century. It does try to trace the philosophy and the mathematics back to personality structure, I think quite successfully and certainly in a thought-provoking way.

But this is a huge and extremely interesting topic to the complexity of which the above comments cannot do but a minimum of justice.

Now, as far as empathy is concerned – I understand Professor Higginson's comment to mean empathy with such human conditions in a mathematical narrative – I think that it comes, as in so many other cases, with understanding. And this, of course, necessitates that we transcend the 'absent-minded professor' stereotype, and see in the grandeur of a purely mathematical viewpoint also its tragedy.

So, (at least for now), a last question for Mr. Doxiadis - Is he happy with the term 'paramathematical'? I should reiterate that I like the idea quite a lot - I'm just afraid that 'para' will be transformed to 'sub' - in the sense of inferiority - by many people. Has he had reactions of this sort from other individuals? Did he consider other possible terms?

As I said, I will not fanatically defend my choice. I am not madly in love with the term 'paramathematics' and I would gladly see it replaced by a better one, if one is proposed. I do see the point about the prefix 'para-' possibly leaving a derogatory aftertaste (due to terms such as 'parapsychology', 'paranormal', 'paramilitary', etc.) and also the point about the more benign uses of it (as in 'paradigm', 'paradise', 'paradox', 'parallel') not fully making up for this. In fact, now that I think of it, I believe that I was conscious of the negative, even the facetious element latent in the term did not escape me when I coined it. But perhaps I purposefully downplayed the concept by my use of name, so as not to anger purist mathematicians by infringing in the Elysian Fields. If the approach contains its own inbuilt warning against taking itself too seriously, I thought, perhaps the EAF mainstream will accept it more easily, reacting by: "Well it's *para*-mathematics, after all. No harm done in the poor guys having their say…" That kind of thing.

Rob Corless

…There is no mention whatever of a central element of any story, namely tension, especially erotic tension. Given the prevalence of love-interest in fiction, its absence from a mathematical quest is a glaring difference that (surely) demands comment.

The idea that mathematics is like a simple detective story is less contentious (but less novel) purely because of the mechanical nature of the genre. But, speaking for myself, when I read a detective story I read it for the characters, the manners, the history, the insights---not for the puzzle (the detective, after all, will solve the problem for me). I love Dorothy Sayers, I love Rex Stout, I love Sara Paretsky, not because they can create puzzles that interest me but because they can create believable people, and put them in tense situations, where self-discovery is often the main point. And these are exactly the non-puzzle aspects, those farthest from mathematics.

I think there is no contradiction, really. I agree that the erotic element is a central aspect of any story -- at least if extended to metaphorical uses of *eros*, as in *passion*, *obsession*, *affection* which need not be erotic per se to provide a story with an emotional powerhouse. And of course character is a fundamental dimension of story. But both the erotic – in the more general sense – and the attraction of character should be there in (para)mathematical stories. (Most people were attracted to *A Beautiful Mind* – which is not even really paramathematical, just a story about a man who happens to be a mathematician -- purely because of a character's suffering and not out of an interest in the early history of game theory.) And if stories of mathematics, in their educational function, are viewed *as concrete stories of human adventure* all the various elements of good storytelling (of which love interest and character are definitely two) are there. But a mathematical story can have great human interest, even if there is no strong direct love element. If we speak about Galois we have it – but what about poor Paul Erdos?

On a more detailed level, I found that there were many contentious statements in the paper; dogmatic, assertive, and (I believe) wrong. "The royal road to a young person's brain [...] is through the heart". It depends on the person.

Of course it does. But unless the person's character is severely impaired – and even then, though in less direct ways – the language of the emotions cannot be ignored when dealing with a developing human being. Show me one young child who becomes obsessed by a brain-teaser and I'll show you a hundred which are profoundly attracted by a fascinating story.

The thirst for hard knowledge can and *does* appear in the young, and appeals to the emotions only cloud the issues. (My italics.)

Indeed, but *craving* for hard knowledge is also partly an emotional phenomenon. The very fact that you use the word 'thirst', a metaphorical word, shows that this is no dry 'need for information'. So, I think that rather than clouding the issues, the appeal to the emotions – again, not as a panacea, but as a valid viewpoint – is necessary. (And if you don't like the word 'emotions', I can say 'the full emotional and cognitive make-up of a developing human being'.) After all, mathematics education has suffered for centuries from a near-total emphasis on the mechanical. And if this wasn't somehow problematic, we wouldn't be here talking about it.

"No expert in---though quite an adept practitioner of[...]" Eh? What's an expert, then? "

I meant I am 'no expert in' in the sense of having no knowledge of the *theoretical* aspects of online investigation, but being a mere experienced user. But I apologize for imprecise word use.

… The central idea of the paper seems to be that story is a good metaphor for higher-level thinking about mathematics (I am not sure what the difference between paramathematics and metamathematics is---what the author calls the EAF model is only a small part of modern mathematics, only about a century old, not millennia as the author claims, and most mathematics happens outside mathematics departments nowadays).

No. One of the central ideas of the paper is that story is a good metaphor for ***some*** of the higher-level thinking about mathematics. As to the difference between paramathematics and metamathematics: paramathematics has been adequately defined, I think: it is narrative, non-rigorous and non-formal. It does not construct theories, in the axiom-hypothesis-proof model, but it discusses them in a narrative mode. On the other hand, metamathematics is *a type of mathematics*, formal, rigorous, abstract, with mathematical theories as its object. As to the first instance of the EAF model – which I agree is not all-pervasive –, it is quite older than a mere century. It is Euclid's *Elements*.

But strategic thinking about mathematics is not, itself, mathematics.

I agree. But 'nonsense creeps in' as you say only if you brand a non-rigorous narrative a rigorous argument. One should know how one is speaking and at what level. And what kind of epistemological mode one is in, every time. And I am against any such thing. Give to Caesar what is Caesar's.

It is quite interesting that this non-EAF mathematics, which is as I stated more nearly the whole of modern mathematics, is closer to the story model than EAF mathematics is.

I agree that non-EAF mathematics (which is most of mathematics *as discovered and as practised* but *not as presented and as taught*) is much closer to the story model – if for no other reason because it often works within the human limitations of redundancy and imprecision, and it is an ongoing quest. Hard to formalize, easy to narrate.

One statement I found quite funny: "A decade ago it was calculated that approximately 200,000 new theorems were published every year. And although I'm sure that almost all of them are right, having been peer-reviewed [...]" Really? I am not even sure that the majority of these theorems have been read carefully even by their authors!

Well, obviously you know more about this than I do, and you also have a better sense of humour. So, I modify my statement, to make it more serious, and I hope acceptable: "And although I'm sure *that many of them are right*".

Quest stories have well-known structures, including things like reversals, frustrations, a mathematical quest of any kind, not just the polished de-scaffolded "Consider X; we assert Y and prove it"?

I am not sure what the question is here, but both the de-scaffolded (I like this term!) and the scaffolded mathematical arguments have story-type structures lurking inside them – both stories (and they are *all* quest stories to the extent that they are all decision-making affairs) and proofs have skeletons of complex possible itineraries and interesting actual routes.

What about "irrelevancies" in stories, as opposed to irrelevancies in story problems or proofs?

Obviously in a story the criteria of relevance can be much wider. An element may be irrelevant plot-wise, but relevant from the point of view of character, atmosphere, tempo – whatever. (Usually, in excellent storytelling, a reader or hearer will not mark anything as irrelevant – everything blends in the world of the story.) In a proof, a real irrelevance can be cut out without any cost to the proof. It is usually an expository excess, not a logical one, if it does not harm the main argument. (If it does, it is more likely a mistake.) But in my argument I am mostly speaking about the plot element of story and it is this element which often mimics the logical unfolding of proof. Paramathematical stories being as a rule based on real stories, their characters are usually given, they are entities to be explored rather than created. And a lot of real 'irrelevancy' may justifiably find its way in a paramathematical narrative, if it serves aspects of the epistemological and/or rhetorical slant.

There are often moral dimensions to stories, which can be examples of behavioural patterns that will enable cultures or memes to survive. Are there moral dimensions to any mathematical fragments? To mathematics as a whole?

A very interesting question. Obviously, looking for an intelligent way to respond to it one would have to consider – among other things -- the sense in which there can be a metaphorical interpretation of a mathematical argument. A naïve and off-the-cuff example: the Fundamental Theorem of Arithmetic could be thus rephrased as "many truths (or problems) can be often reduced to much simpler ones". I think that doing this transposition, to increase the interplay between mathematics and real life, makes a lot of sense occasionally. And in this sense there is a two-way interaction between mathematics and real life, structural similarities often emerging via metaphor or, in reverse, abstraction. This is also partly true of the relationship of stories to proof. Of course, to go a step further and speak of the 'moral' dimension of a mathematical truth, one would need to refer to a set of values, and this would be quite impossible inside mathematics. But an 'isomorphism' into a world with values, would make it less so.

Glenn Gordon Smith

In your paper, you refer to three levels of chess stories with analogs in mathematical story-telling (tactical, strategic, cognitive/psychological/ historical).

But how should the cognitive/psychological/historical level be further broken down into smaller categories for an even more productive analysis?

This is a good opportunity to clarify the argument about the story-proof relationship. As mentioned already, the three levels refer really to three characteristic points in a curve. Let's take the case of chess: at the one, low-complexity, end, we have an analysis that is purely combinatorial in nature (a pawn and king ending, say) of which any 'story version' would be totally uneconomical and unenlightening. (One has to remember here that when telling stories – and this point was raised in many of the c & q's – we are *not* speaking purely formally and with merely literal truth in mind. A story operates at least partly metaphorically (that is why *The Old Man and the Sea* is not of interest only to fishermen) and its deeper levels come from – and are addressed to – human beings, not logical machines.) Whatever meaning lurks in a purely combinatorial simple chess situation is ideally expressible practically only in the 'algebraic' language of chess players. But a situation becomes more complex, strategic considerations begin to prevail. Thus it is meaningful to give a general guideline for opening play of the type "a player must try and control as much as possible the centre of the board and develop his/her pieces as much as possible". Obviously, this is a general truth, that *can* often be contradicted, in specific situations, by tactical (or combinatorial) considerations. Yet, its generality brings it much closer to a narrative description of a situation, and a non-rigorous, metaphor-laden language, that comes from- and appeals to- levels of the mind beyond the formal-combinatorial is meaningful here. As we go up, psychological and other factors also become more to the point.

It is pertinent to quote here chess Grandmaster and writer Genna Sosonko: "However, there is a great difference between analysis and the actual process of playing. A game of chess is not a theorem, and the one who wins is by no means always the most logical and consistent, but often the one with the greatest endurance, the one who is the most practical, clever, or simply lucky." (p. 74, *Russian Silhouettes,* New in Chess Editions, Alkmaar, Holland, 2001). Obviously, concepts such as 'endurance', 'practical', 'cleverness' or 'luck' are much more amenable to a narrative than a formal language. So, what really happens is that at any level, a game or a more general conflict, a description of a game that seeks to capture all its depth must contain both tactical-combinatorial and narrative elements and it is really the relative prevalence of the one or the other that makes me speak of three levels, schematically. But there are really many more.

As an aside here, it is interesting to note the various ways – *levels*, really – at which many important players annotate their own, or others', games: again you may get everything from tactical possibilities and variations – these sometimes discovered *after* the game – to purely psychological or practical considerations. The annotations of ex-World Champion Mikhail Tal are good examples of such multi-layered expansions of the game, often rich in narrative elements. But of course, the narrative element becomes more important if we leave the level of one game and go into a player's overall performance over a period of time. Chess players often study the biographies (and not just the games) of the great players, to learn more about the game.

I like the issue of the 'soliloquy' raised by Dr. Smith very much. In fact, many Shakespearean soliloquies (and 'To be or not to be', the one he mentions, eminently so) are just stops in the action that recapitulate a protagonist's alternatives, and as such are very reminiscent of both mathematical and chess thinking.

But when talking about the thought processes of chess players we must be very careful. A lot of the study of the way of thinking of chess players was historically motivated by the attempt to formalize this process, even before the attempt to model it on a computer. And the study by Adriaan De Groot mentioned is indeed a pioneering study in this direction. But it is important that it has the aim *to show the extent to which chess thinking is a purely logical process*. And an even poor player, like myself, learns to apply this four-step process in analysing situations on the board – but not always successfully, and not always period. There are cases where it is useless.

Yet the study of chess psychology has gone a long way beyond De Groot in understanding how chess thinking *in human beings* also differs from such a formalism. For example, it is known that top player levels have an extremely developed memory and an internalised alphabet of basic game-patterns or sub-patterns (for a grandmaster this can be in the order of tens of thousands) and, as a rule

– this came as a big surprise -- they do not calculate more than average players. Of course, they may *calculate* (this is the word used in chess) at times, and in certain situations. But in others they may be thinking purely strategically, intuitively and even purely psychologically during a game – in fact, the great mathematician and ex-World Champion Emanuel Lasker was known as a very subtle psychological trickster on the board, often executing 'bad' moves that he thought would be most effective at that moment for that particular opponent. (An excellent, if slightly dated review of existing research and ideas is *The Psychology of Chess* by W. R. Harston and P. C. Wason, Facts on File Publications, New York, 1983; also important is Pertti Saariluoma's *Chess Players's Thinking, A cognitive psychological approach*, Routledge, London, 1995.)

The soliloquy, voicing ones thoughts while making a pivotal decision ("To be or not to be?"), is fundamental to literature and holds great drama if consequences are terrible or triumphant. How is this building block of narrative translated to the mathematical story? Through think-aloud protocol of expert problem solving? Also, in order to make individual mathematical problem-solving compelling as a story, how does an individual solving a mathematical problem have terrible or triumphant consequences? What would be examples?

In a mathematical story choices can be 'low-level', i.e. combinatorial and in almost direct rendering of the underlying mathematical argument, but also much more complex, at the 'higher' end involving purely non-mathematical concepts such as persistence, endurance, will-power, self-doubt, obstinacy, despair. I can easily imagine such 'soliloquies' enriching a description of the mental and psychic turmoil Andrew Wiles went through in his search for the proof of Fermat's Last Theorem. And the terrible or triumphant consequences are very visible there – at the personal level. Also, if I may, I will mention as example my novel *Uncle Petros and Goldbach's Conjecture* which contains exactly such soliloquies, as representing the thoughts and the dilemmas of the protagonist, on which road to take next in his mathematical odyssey.

But such questionings, of self or other, can operate at all levels. When Alice asks the Cheshire Cat which way to go (a low-level, combinatorial piece of advice) the Cat asks where she wants to go – a strategic question. And when Alice says it does not matter where she goes, she gives her a higher level, meta-answer, saying that then it doesn't matter which way she goes!

Immaculate Namukasa

Would he identify the series of books "Sir Cumference and... adventures", by Cindy Neuschwander to be good mathematics story books? How and where is paramathematics happening in classroom? I guess I am asking for classroom examples.

I have read three Sir Cumference stories with my five-year-old son and he enjoyed them very much – a good sign! Now as to the extent to which these are good *mathematics* story books: well, they are certainly useful in setting abstract concepts in a human context; and indeed they are 'stories of problems'. However, we must keep in mind that these are definitely tales where the rhetorical-didactic element predominates, as they certainly do not add depth to the mathematics in question. And although the mathematics involved, especially in a more clear-cut 'story of a problem' tale, like *Sir Cumference and the Dragon of Pi,* is very close to level 1, the tale does have importance as depicting mathematical method in a way that can make it emotionally interesting. But one does have to keep in mind the age they are addressed to. At that level, the narrative approach of 'making interesting' through a simple idiom works. But it is hard to make this kind of simple, fairytale idiom work for a higher age group.

On page 24 he talks about three levels of mathematical engagement – syllogism, strategic and Narrative levels, how do these relate to each other? What diagram or metaphor, if any, does he use to illuminate this relation? Would he use a diagram like the one on page 6?

This has already been referred to. In essence the level depends on the *extent to which a narrative description of a part of a process – rather than a formal -- could be interesting.* At level 1 this is practically nil, but it rises as we go up. This could of course schematically be represented as an x-y curve, with the level (y) being a function of the importance of a narrative rendering of a mathematical process.

Obviously, in a *completed* mathematical process (i.e. a peer-reviewed-and-accepted proof) as in a chess game (or a match, i.e. a series of games) that has *already* been played this can always be described completely formally, if we want – it is theoretically possible to do so. But we talk of a higher level process when a description incorporating narrative is meaningful. But, talking chess for a moment, a purely formal rendering and line of thought cannot be given for a game in progress: too many unknowns. In any mathematical investigation or game (s) the level rises as narrative elements – the more the higher – can add significance to the formal rendition. Of course, schematizing – and, even more, trying to quantify this process – is only done using the language of analogy. The important functions of the narrative process are mostly those that cannot be formalized, referring as they do to higher levels of human functioning.

Liz deFreitas

If our focus is on: "how storytelling can help create good mathematics?" will we impose an instrumentalist vision onto narrative and possibly diminish the power of the text? Similarly, metaphors that are explicitly didactic can give too much closure to an art form. How do we "use" narrative without killing off its power to reach the reader?

Again it depends on the level – and, in fact, this is how the whole issue of levels arose. My point about levels is that there are only some parts of a mathematical process that can be non-trivially rendered by narrative. If the right level is addressed, this should not end up with trivial narratives. Practically speaking, I think the answer here is to work from real-life, historical and/or mathematical stories. It is in these that both the mathematics and the narrative can remain interesting at higher levels of sophistication. The Sir Cumference type story can *not*, as a rule, be meaningfully extended to higher levels of mathematical sophistication.

There are no formulas for narrative (despite what some how-to manuals suggest) and much of our engagement as readers comes from the play of the language. With regard to the intersection between story and math, is this "play" confined to what you have called "paramathematics"?

I am afraid I do not exactly understand this question. But I disagree with the generality of the statement 'there are no formulas for narrative'. Indeed, there are no general-use formulas (as, especially, some 'how-to' Hollywood screenwriting books suggest – Syd Field seems to have started this and the approach, and its latest guru is Robert McKee) guaranteeing excellent results. But there is a long tradition, going all the way back to Aristotle, and progressing to us via Vladimir Propp, the Russian Formalists and the various taxonomies of literary genres (Northrop Frye's 'archetypes of fiction', the Aarne-Thompson classification of fairytale types, etc.) of a more formal study of fiction, where the identification of patterns and underlying flow chart- or graph-like plot structures does yield some significant insights. As a writer, I wholly agree with the point about the importance of language in a narrative. But as Dr. deFreitas says "*much* of the engagement as readers comes from the play of language" (my italics). Yes. *Much* – but not all. Although language is an important element, plot, character, theme – things that can be found in stories of mathematical history and biography – are also very dominant. And I do not believe that in good storytelling the medium *is* the message. It is part of the message, certainly, and it helps us get the message across more profoundly and effectively. But there are extra-linguistic elements in good narrative and it is these I principally address in my discussion.

Margaret Sinclair

You comment that one "can only be rigorous at the formal (the lowest, in terms of complexity) level of knowledge". Could you discuss how the formal relates to early experiences in mathematics, i.e., are there some aspects that are foundational? Is it necessary? Do we build narrative on rigor, or can we/should we build rigor through narrative?

Well, to begin with I must say that most of what I know about early experiences in mathematics derives either from my own, or my children's education – so, do not expect anything profoundly well-informed from me on the side of experience. But as with my discussion of levels, I would say that formalism and rigor are not all-or-nothing phenomena. Obviously, *Sir Cumference and the story of Pi* is not at all formal, while Whitehead and Russell's *Principia Mathematica* is extremely so – but there is a whole range in between. If we look at what we mean when we wonder when a young child's mathematics are rigorous and/or formal, we obviously do not mean the same as we would with a college student of mathematics. A proof produced by a math undergraduate that would be considered sloppy or relying too much on intuition, would be a miracle of rigor for a ten-year-old.

So, what can rigor and/or or formalism mean for a five-, a six-, a seven-year-old? Obviously, at the meaningful-for-this-age point of the low-high rigor scale, I think it would mean something like some dose of precision and abstraction. The demand for higher precision is something students meet increasingly as they grow in must subjects (e.g. history, grammar, geography…) while abstraction is slightly different in mathematics: a grammatical tense takes a certain form and Paris is the capital of France *because that's the way things are* – in these subjects, as in mathematics, it is precise knowledge that is imparted (and insisted on). But in mathematics abstraction goes back to cause. I.e., although we do not teach young children the Peano axioms for arithmetic, and we may encourage them to memorize the multiplication table for reasons of efficiency, only a very insensitive teacher would say to a flabbergasted youngster that 3x5=15 because that's the way things are, or God made it so, rather than explain that what this in fact means is that 5+5+5, i.e. adding 5 three times is equal to 15, something the child can verify and comprehend from the simpler rules of addition.

So, as a rule, I would say that demands for rigor and precision are (and should be) very elementary for young children, and increase with age. And as abstraction in some of its meanings or other is probably the most common cause of dislike of mathematics, obviously the less traumatically, and the more gradually, one introduces it the better.

But, interestingly, children do not have a problem accepting formal rules, sometimes even totally absurd ones, in the context of a new game, and this should make us think. The reason probably has something to do with the fact that a game creates a whole mini-world, which is through its story background and/or competitive element much more wholly significant to a child. And the story context, even with very young children, could help them accept the abstract (formal) rules of mathematics much more easily, especially in the context of games. I know that this is also the opinion of Howard Gardner, as mentioned – if I remember correctly, unfortunately I do not have access to the book at this moment – in the last chapter of his book *Extraordinary Minds.*

I know there are many approaches to this, but a narrative approach to problem solving (which might have quite a lot to do with an algorithmic one, relying on the story-quest analogy) would be very beneficial for young children. And this does not mean pure storytelling necessarily but creating story context – as in a game, for example.

Going back to my discussion of levels, I want to remind you that the narrative element becomes less interesting and effective if we go into levels of mathematics where rigor demands (in the childhood sense) are higher. Thus, there is no point in teaching the way to find the square root of a number through a story. This, equivalently, would be like teaching a chess student how to mate with a rook and a bishop. Of course one could say 'first you cut off the enemy king's ways of retreat' (a statement that is also 'narrative') but one would assume that a player at the level where he/she can learn this would need no metaphorical ballast to understand it, but would merely apply it on the chessboard.

But if we are talking about a general approach to problem solving, storytelling can be very pertinent. For example, one can see stories that would embody Polya's principles (listed on page 5 of my paper) on problem-solving very well.

Although talking about 'building narrative through rigor' or 'rigor through narrative' is a little bit like comparing apples and oranges, I have to say that rigor and narrative are *not* opposites. Just different things. There is no reason while rigorous rules and arguments cannot help in the creation of narratives – *to a certain extent,* of course – and there is really no reason why narratives could not teach rigor. The exemplary mode is as much a part of fiction, in either direct or indirect forms, as is the cautionary. And one cannot dismiss literature which is close to the exemplary or the cautionary as trivial and didactic. *The Brothers Karamazov*, as most of Dostoyevsky, contain strong elements of both, and *Othello* or Mann's *Doctor*

Faustus, are very great literature, despite being very strongly built on the cautionary template.

Present teachers have been very successful, moderately successful, or unsuccessful at the EAF model. For each group what do you see as the critical interventions that would enable them to understand/adopt elements of the paramathematical field?

I can say, to start with, that I am pretty sure that the EAF model should not be *forced* on children. Obviously, some children would react better to it than others. (And a mildly autistic ten-year-old might love it -- see the *Curious Incident of the Dog in the Nighttime.*)

Here I want to remind you of Von Neumann's brilliant observation, which is highly significant (if not wholly true), a wise aphorism more than a rigorous truth: "In mathematics you don't understand things. You just get used to them." I think that the truth behind this comment has to do with the fact that the abstract mode is foreign to human ways of knowing and learning and thus needs getting used-to. Of course, I'm sure many mathematicians believe that they *understand* at least certain things in their fields -- but, again, Von Neumann seems to be saying that this 'understanding' is really nothing but the familiarization with the laws of operation of a foreign universe, and this needs time, increased familiarity, experience, co-habitation, the diminution of the initial fear of the strange and incomprehensible… I think it is the same with children. But while a mathematician exploring a new sub-field (or a pre-med student needing to advance in calculus to get in a good medical school) has an increased motivation to do so, probably extraneous to the field itself, a young child usually has no reason to spend the quality time necessary to "get used to mathematics". It is precisely in this that the story context is helpful. It would be a very unusual 7-year-old child indeed who, given a choice of listening to a story from a good storyteller (or seeing a favorite DVD) or doing two-figure sums, would systematically choose the latter. Stories are cognitively much closer to life-as-lived than the abstraction of mathematics. This is an obvious point though. What is not so obvious is that mathematics and stories mix.

I think that we must try and see how much of 'Haeckel's Law', from 19th century biology, is applicable to paramathematical thinking. Haeckel said that "ontogeny recapitulates phylogeny", i.e. the growth stages of an organism, from one cell to full growth in some sense imitate the phylogenetic origin of the particular species to which it belongs. I do not how true contemporary biologists consider this in biology. But obviously it is *to a certain extent* true in mathematics: the stages of mathematical understanding of a growing human being in some way mirror the evolution of mathematical science – I stress that this is useful only as an inspiration, not as an absolute truth. But if taken with an adequate amount of grains of salt, this can be a very interesting guideline for teaching. In fact, it is interesting to note that the whole 'New Math' affair was based on a gross violation of exactly this principle: people starting to teach children mathematics with what only very recently in historical terms had been thought to be, by the Bourbakists mostly, the true foundational core of all mathematics, i.e. set theory.

To the other end of the EAF model of mathematics stands the heuristic, messy, calculational, applied one. But the strength of the paramathematical approach is that speaking as it does *of the historical adventure of mathematics as created by people* it can address both models – and both become humanly understandable if viewed within the human story that created them. In fact *Logicomix*, the graphic novel I am working on at this moment with theoretical computer scientist Christos Papadimitriou is the first (I think!) paramathematical comic book, and recounts, in its first part, the story of the foundational crisis of mathematics, starting from Cantor and ending with Godel's Incompleteness Theorem and Turing's first results. In other words, we use the story medium to speak of the shortcomings of ultra-formalist demands. Of course, as the comics medium is very reader-friendly, the emphasis here is more rhetorical; but the epistemological slant is not at all absent – in fact it contains quite a sophisticated narrative exploration of the relationship of logic to madness.

Donna Kotsopoulos

The discussion paper prompted me to reread Sfard's (1998) article "The Many Faces of Mathematics: Do Mathematicians and Researchers in Mathematics Education

Speak About the Same Thing?" Professional identities, mathematician versus researcher, often form the basis for the valorisation of a specific understanding of "good mathematics".

You are right. However, as there is no way to discuss valorisation of mathematics formally, we either must exclude such discussions from the world of mathematics – which no one does, anyway – or take recourse to another language, one taking account history, to speak intelligently about it. Of course, the problems of valorisation are not necessarily ultimately 'solvable'. But most of life is not, anyway, and mathematics, when we are not talking from inside an EAF viewpoint is a part of life.

The discussion paper details three levels of narrative inquiry in mathematics: tactical-cognitive/psychological/historical. With rigor only being a contributing factor in the tactical-computational level. For the mathematician, a formalist, rigour is paramount. **If mathematical narrative is to be understood as a form of non-trivial thought then how can professional identities that associate "good mathematics" reconcile rigor as being only an elementary component of paramathematics? Would rigor not underscore all three levels? How are the latter levels of narrative inquiry in mathematics assessed?**

Rigor exists in all three levels. But in the higher levels it does not dominate. This is not an issue, however, of doing 'rigorless' mathematics – it is talking about the sides of issues that are not totally dependent, *at that level of discourse*, on rigor. To give an example, let us assume that we discuss the use of computers in problem solving and we tell the tale of the Four Color Problem also taking into account the Appel-Haken solution, to which many EAF mathematicians will not give the distinction of being a proof. Of course it is rigorous – as any proof has to be, in fact it is *too* rigorous, so rigorous that human beings cannot check it within any reasonable amount of time. The questions related to whether this indeed *is* a proof, as well as what the story of the Four Color Problem means for how we do and understand mathematics are best told in narrative form, to understand their full complexity. No field of mathematics or metamathematics is equipped to answer the question of whether the Appel-Haken proof is a… proof. But this must be discussed –

Paramathematically, I believe. (Robin Wilson's book *Four Colours Suffice* goes some way in this direction.)

At another level, try to imagine a book (not yet written, to my knowledge) introducing topology to adolescents, but going beyond the usual donut/pretzel, Möbius strip level. A book referring both to general topology, problems of continuity, the foundational problems those created in analysis, how they were solved in the nineteenth century by Weierstrass and others, then geometric topology, even ideas from algebraic topology, all the way to some understandable low-level truths from advanced modern topology and also applications into problems of space arising out of relativity. Such a book, where the narrative thrust is dominated by the need to know arising from specific problems and crises of knowledge, both intra- and extra-mathematical (as well as intra-domain or intra-psychic in some of its practitioners!) could well give 14-16 year olds a very good idea of how a mathematical field progresses, outlining many of the methodological issues, showing graphs of the way that knowledge progresses, how we generalize from the known, how we form hypotheses, how proofs can be wrong, and so on. All this need not contain but a minimum of formally perfect complete processes or proofs and could often use very imprecise (vis-à-vis rigor) statements. But it would give, if well written, an admirable sense of how mathematics works, and why, and contain a good methodological background. And if there are exercises, they would not have to be topological at all. They might be logical puzzles, generalized logical and methodological techniques used. This would definitely be a non-formal book. But it would be a mighty interesting and useful one.

Sonja Rowhani

I only have a very rudimentary knowledge of chess – maybe comparable to some students' knowledge of mathematics. The conclusions you draw from the three chess stories are diametrically opposed to the conclusions I would draw from them. As someone trying to learn chess, a detailed, tactical story may be more useful than a higher-level story. While for an expert the higher level story may serve as a cognitive tool, it may confuse novices. It seems that the way we

read/understand stories is very personal limited by our own understanding. What is your perspective on this?

A beginning player that is introduced to chess by an experienced player-coach, will be given equal doses of tactical-combinatorial *and* strategic lessons even from the very beginning. (Strategic lessons are much more general and less formal.) Tactics and 'calculating' offer very near-sighted views of any one point of a game, and as we know to get somewhere it does not suffice to be able to see, or even to have a map, but w*e must know where we are going*. Only strategy can do this – and believe me, it is very different from tactics. The more a beginner gets into the world of competitive playing, the more the third level, of a historical and more general understanding of what they are doing becomes necessary to progress. For example, once a player has grasped the rudiments and can play competitively at some decent level, he or she will have begun to acquire and demonstrate a particular *style*, which is very obvious to a more experienced player. Once at this level, it is extremely valuable to study the way great players with a similar style progressed – not just played in specific games – in their chess careers. Clearly, a lot of chess is purely combinatorial. But not all is.

To me it seems that stories in mathematics could succeed in motivating the student and creating an interest in or appreciation for mathematics more so than teaching methods and tools needed to tackle mathematical problems. For example, I am an avid mystery book reader. However, even after reading many stories, I don't think I am qualified or able to solve a crime. Could you delineate the place of stories in mathematics education as you perceive it?

You are right that motivation is one of the primary motives. But I don't agree with you on the uselessness of mystery books in solving a crime. Of course, it would depend on the sub-genre to which the books belong. No amounts of Agatha Christie-reading will make you a Poirot, and that is because they are so little realistic, both from the practical and the psychological viewpoint. But reading good police procedurals and, say, the Inspector Wexford novels of Ruth Rendell, or suchlike more down-to-earth whodunits, would get you very much closer to the mind of an investigator than a complete innocent. And, should you care to, it would teach you quite a lot about investigating and solving some types of real-life problems – not necessarily related to homicide! But, primarily, do not forget that those books are written to entertain. Paramathematical books, as I define them, are not. This is not their primary aim. They are created to investigate mathematical issues and/or to instruct, i.e. with the epistemological and/or rhetorical slants I referred to earlier. Of course, *A Beautiful Mind* or *Good Will Hunting* or *Proof* will teach you nothing about mathematics. But they neither can nor want to. But they are not paramathematical, they are not motivated by a need to understand mathematics, as *Who Killed Roger Ackroyd* is not motivated by a need to understand investigative procedure. But view the full 'CSI' tv-series and read the Patricia Cornwell mysteries… Would you then say that you have not learned a good deal more than the next person about criminology and laboratory investigation?

George Gadanidis

I find the following comment interesting:

"The reason I will not be talking much about education is because I believe that how we teach mathematics, as a culture, is shaped by how we do mathematics."

Papert in *Mindstorms* says that children enter school as enthusiastic, curious and capable mathematical thinkers – they have to *learn* to be otherwise. Yet how *they* do mathematics is typically ignored (and not reflected) in how we teach mathematics. In fact, some studies comparing the mathematical thinking of grade 2 children with that of grade 4 and grade 5 children (doing identical mathematical tasks) have shown that their mathematical thinking deteriorates, relying more on procedures they do not understand (Kamii; Reid) So the 'we' in the above quote probably does not take them into account. **How might the mathematical thinking of young children (before it is influenced by school or 'well-meaning' adults) help us answer your question of 'what is good mathematics'? Do you see part of the answer in your children?**

Obviously, a question referring to value, such as "what is good mathematics" depends very much on context. And I agree that in the context of education, "what is good mathematics for children" needs to be taken into account – at least to an extent. I definitely see, with my own children, that if you do not want to alienate them, mostly at the pre-school age, it's better to let them do things (sums, for example) their own way, than sternly impose a method.

But regarding the story approach, creating rich storied contexts for, say, 2-6 year-olds, can unleash a lot of their own creative thinking and potential – more so than a pre-arranged learning conceptual environment. Of course, in mathematics especially, this type of freedom has to be regulated occasionally, as a lot of mathematics teaching, especially in its emphasis on abstraction and preciseness – I have learned from a previous comment not to use 'rigor' uncritically, when talking of children – goes against the natural, free, combinatorial methods of the storyteller in the child.

Also, I wonder if the reverse is also (or more) true: 'how we do mathematics is shaped by how we teach mathematics.' (A chicken-egg problem?)

Well, I would say undoubtedly it is a chicken-egg situation. The re-enforcement of the EAF model in mathematical practice certainly comes from EAF-inspired teaching. And, now that you mention it, perhaps it is no accident that the ultra-EAFist Nicholas Bourbaki approach originated in France, a country much admired for the extremely high, purist standards of it mathematical education, especially in the École Normale, from which most early Bourbakists came! Good point! And as with any chicken-egg situation, you can only start to change it at the level of the chicken, it being much more flexible than the egg! I mean, in this case, by intervening in the education, more so than the practice – which mathematicians wouldn't let you, anyway!

I wonder what would be left if 'story' is taken out of human communication about things that matter. Would we be left with recipes?

I do not think we would be left with anything – it would be impossibility. Or, if anything at all survived, we would be left with objects. You see, I believe story is inextricable from our fundamental mechanism for conceiving of the flow of the events in our world in a non-chaotic way. The symbolic-linguistic capacity which we possess, as animals don't, allows us to conceive of and express the world as series of meaningful actions. And in this sense a recipe is ALSO a story, and this is important for our argument.

Peter Taylor (Queen's University) many years ago visited Bill Barnes' poetry class and noticed that Barnes brought to his class poems he was passionate about. On the other hand Taylor brought mathematics to his first year Calculus class that did not interest him in the least. Eventually, Taylor and Barnes designed their Math and Poetry course, which they co-taught, and where Taylor brought to class only mathematics that he was passionate about (his mathematical poems).

Oh, yes, three cheers for subjects we are passionate about – in any case! And of course passion has to do with the emotions (a gross tautological understatement this!) and the emotions are important and stories are ideal carriers for emotions. But because we cannot expect, except in a very utopian environment, the school syllabus to be molded by the teachers' passions, it is good to have stories around anyway, hoping that they embody – if not the teachers' – then at least their creators passions. And good stories certainly do that and have the advantage that the passion is visible and communicable, whereas that is not always so in good mathematics. Newton certainly must have felt passionate about the calculus. But that cannot be seen in the definition of the derivative.

It seems to me that at the heart of a good math story is a good math problem. But what is a good math problem? I realize that it much depends on how the problems are lived (how they come to life in the classroom), but some problems have more potential than others for leading to a good math story. A simple example in primary school would be the contrast between "what's the answer of 2+2?" vs "the answer is 4 – what was the question?"

So, what in your mind is a good math problem? That is, not what is good mathematics, but what problems lead (or are more likely to lead) to good mathematics?

Perhaps because of my relative inexperience with these ideas, I tend to distinguish two kinds of paramathematical stories – whereas deep down they are one. The two kinds are those of a created quest environment (for younger children) and those where the quest environment comes from the world and history of mathematics itself. And although I am a parent (three times over) I have very little experience of teaching kids – so I have thought much more and can talk much more intelligently (I

hope!) about the second case, the paramathematical approach to the world of mathematics, which would definitely not be ideal for very young children.
(The NUMBER DEVIL by Hans Magnus Enzensberger is obviously a good case in point, for the first category.) But, as I said, this is perhaps due to my inexperience in the sense that the same rules apply in both cases. But a good story is a good story – period. Thus, if you are looking for the mathematics that would create a good story, you should look for the mathematics that would support a story with interesting characters, fascinating dilemmas, an engaging story world, strong reversals of fortune – what you would look for in any story.

My only warning to anyone attempting to write and or design such stories and environments would be to be aware of the 'gear-shift' factor which I mention in my comments to Kamran Sedig, below.

I wonder about the following statement in the context of the education of young children:

When I define paramathematics as, mostly, the 'stories of problems', I do not mean that any account of the story of a problem is of intrinsic value to mathematics. A book such as Singh's on Fermat's Last Theorem, though a great read and a fascinating introduction for the non-mathematical public to a famous problem, does not in any way shed additional light to it. Yet other books – the list is not exhaustive – like Martin Davis's *The Universal Compute*r, Peter Pesic's *Abel's Proo*f, Amir R. Alexander's *Geometrical Landscapes* and Karl Sabbagh's *The Riemann Hypothesi*s, contribute I believe, if not to the advancement then at least to the deepening of our knowledge of mathematics by telling the stories of problems in an interesting way and adding sophisticated 'para-'syllogisms to the formal development. It is worth noting that the author of the last of these books – incidentally republished two years after its original publication as *Dr. Riemann's Zeros* (!!!) – is a documentary producer who studied anthropology and employs in his interviews the approach of a field anthropologist trying to discover the *Weltanschauung* of mathematicians. So: paramathematics is not just for mathematicians, manqué or otherwise!"

I wonder how we do this for young children, without losing sight of the mathematics? It's hard to imagine engaging them with stories about the

mathematics of mathematicians. Do we need to write stories about children's mathematics (see also question 1)?
What would work for them? Perhaps considering what math stories you would want to tell your children might be a way of answering this.

Ay, there's the rub! As I said earlier, I have not thought so much about this, as about the 'stories about mathematicians', which can certainly be made fascinating for adolescents. Let me give you some very off the cuff ideas on working with younger students.

a. It is more likely that they should be stories of quests in fictional environments, rather than the world of mathematics.

b. Still, the world of mathematics and mathematicians should be gently introduced, the figure of a mathematician as a fictional, or half-fictional hero.

c. Because of the problem with 'gear shifts' (i.e. people liking to know what they are reading/what genre they are involved in and not liking to be radically subverted) I would not include too much 'low level' computational stuff in a higher level story. But a rather intricate storytelling sequence can be made solely at the low-level. Rule of thumb: if you want to address 'low-level' issues, make the story line and environment very lean. If you want a rich story context, don't do too much low level work.

d. It's always a good idea (and whodunits are the model for this) to assimilate methodological and epistemological knowledge into the story – it is much easier to do, seamlessly, than the mathematics. In addition to using stories for teaching skills like sums, operations, fractions, etc, try and do 'mathematical' stories for early-age students which are really logical, i.e. they teach them the methodology and tasks of a logical search. Sherlock Holmes is always referring to his methods and epistemology, there is no reason why we could not do this with logic and children. (Some of Smullyan's books are great in this, and especially his chess books – as models.) Try to do stories that make of a child a logician – and perhaps show how that can be applied to mathematics, occasionally and softly. A good hero: St. Basil's Hound – the one who invented the reduction ad absurdum!

To summarize, and to exaggerate the argument somewhat, let's consider "Where does mathematics come from?" This is a question we discussed in last year's symposium, in the context of Dissanayke's "*Homo Aestheticus: Where Art Comes From And Why*." Dissanayake presents elaboration as a process of making special, noticeable. She states, "I suggest then that "elaboration" (another word for "art") is a human need, and that humans evolved to need to be able to show their regard for things that are important to them, and show it *artfully*." Perhaps 'elaboration' may be extended to include the human desire of making the self more complex, of seeing and shaping experience in new and more sophisticated ways. This does not mean that the art or mathematics created is necessarily more complex, rather that it captures or depicts a greater complexity. Thus, striving to achieve greater complexity is normal, natural and necessary for a human being. In fact, this striving for greater complexity, in art, in mathematics, in science, and in many other areas, is an obvious trend in human history. To use Dissanayake's terms, elaboration is normal, natural and necessary, and art and mathematics are expressions – instances – of the human desire to elaborate.

Interestingly, Dissanayake does not look for the answer to "Where Art Come From" in artists or museum art. Is paramathematics overly concerned with the mathematics of mathematicians? Where else might we see mathematics as "normal, natural and necessary"?

I think the analogy is interesting. I am an avid fan of Dissanayake, when she talks – as she almost always does – about art. But I think mathematics is slightly different. It (may) be 'normal, natural and necessary' at a low level, a level where the complexities of mathematics and logic (and thus the complexities of the human adventure lurking behind them) are rather simple. The problem is that it is rarely fascinating at that level – unlike art, where a simple sonnet, Beethoven's *Fur Elise*, a Mayan sculpture or an Italian fairytale can all be great art. But not so in mathematics. There is no poetry in simple fractions or sums, or the Pythagorean – or if there is, again it is of the 'Mary had a little lamb' variety, comparatively speaking. (But even simple logical puzzles can be fascinating – and logic is a part of mathematics!!!!) So, we need stories to make accessible to the soul what is mentally complex, to create that bridge. The naturalness of mathematics (to the small extent that it exists) is useful at a very low level. If we want to show the naturalness of logic, as a part of the mathematical process, that may be simpler: stories are more logic-friendly than math-friendly.

Kamran Sedig

I quite agree with what you say about the three levels of stories (tactical, strategic, meta/grand strategy). For a number of years I have been wondering how to create a connection between all three levels for the purposes of interaction with mathematical representations and their exploration**. Do you think all three levels can somehow be represented simultaneously so learners can explore them together and find out how they are related? If so, do you have any thoughts on how this possibility can be operationalized using online mathematical investigation tools?**

Let me for a moment relate my experience while writing my book *Uncle Petros and Goldbach's Conjecture*. Of course, this was addressed to readers of fiction, and I did *not* assume the average reader would be interested in mathematics – quite the contrary, in fact. (But that is also a good assumption if we are addressing the average primary or secondary school student!) What I did want to have there – without yet being conscious of the precise notion of levels – was really to have all three. Starting from the top, I definitely wanted to show how history (Gödel's theorem), society (the unwritten laws of academia), psychology (Petros's character) interacted with the mind of a researcher and how they influenced his career. For that narrative suited me perfectly, and a lot of the positive reactions to the book from the mathematical world have been variations on the theme 'manages to show how a research mathematician works', etc. And I think this is interesting, both as an epistemological and rhetorical task. Going to the middle (strategic) level, I again tried, I think somewhat successfully, to give an idea of the type of thinking involved when a mathematician tackles a problem, e.g. by using the reductio, by simplifying the problem, by looking at it from a particular side with more developed techniques (the partition problem) etc. And again, this could be done more or less 'narratively', where a good part of this could be the internalised, 'soliloquy' type of approach mentioned by Dr. Smith. As for the lowest level, the 'tactical', or more directly computational, I thought there would not be much sense in putting in stuff of this kind. I had an instinctual sense that if it can't be 'narrativized', then it stays out. Thus, although, for example, I included at first the full proof of Euclid's infinity of the primes, as Petros explained it to the nephew (it's just three or four lines of course) I then cut it out – along with some similar stuff.

But had I been addressing a reader more benevolently inclined to mathematics – say, in a non-fiction work of Singh's *Fermat's Last Theorem* variety --, I would of course go a bit further.

Now, speaking as a writer, I know that the reader's intention in picking up a book is crucial. And though I try to be reader-friendly at all times, and would not be interested in addressing a reader with the declared perversion of liking "difficult books" in literature (i.e. a self-proclaimed masochist) I of course understand that the author of a mathematical treatise can expect a bit more of a willingness to take pains from a reader than the author of a commercial 'page-turner'. And a paramathematical text does not really as a rule expect to succeed, as a rule, by springing on an unsuspecting tourist looking for a long flight's fun read and enslaving his heart to the Queen of the Sciences.

Still, how much 'low-level' arguments can we put into a narrative text? To me, as a storyteller, the operative word here would be 'seamless'. If you are writing fiction (of the not intentionally 'difficult' variety) you do not like your reader to have the feeling he/she is changing gears all the time – or if so, you look out for abrupt changes. If they aim at shock value, that may be alright, but as a rule you go for an underlying unity of style. Thus, although you can use two narratives, a present-tense and a flash-back, and move quite easily from one to the other once the convention is established or, again, you can use two very distinct and different narrative voices (say a university professor and an illiterate and not-very-bright wino) and again achieve an easy co-existence, you cannot easily say "now you are reading a story / now you are solving an equation / now you are reading a story / now you are doing an integral". The two don't mix very well – even if the reader is fluent in both literary and mathematical reading. In fact, that is one of the main problems, in my mind, of that old classic, E. T. Bell's *Men of mathematics* – when I was in my math reading mood, I'd rather read more hard-core (and more up-to-date, symbolism-wise) versions of the achievements of his heroes, and when I was in my narrative-thirst, hero-worshipful mood I found the mathematics distracting.

Thus, in conclusion, I'd say that paramathematical thinking and writing should tend to stay closer to levels two or three – trying to leave level one, unless for the purposes of illustration or explanation, if not alright out, then at a marginal level.

But some excellent paramathematical works, such as Peter Tasic's *Abel's Proof* (also of great interest is his *Mathematics and the Roots of Postmodern Thought*) Leo Corry's wonderful *The Origins of Eternal Truth in Mathematics: Hilbert to Bourbaki and Beyond* (available on the internet, from his web page), Alain Connes two books of conversations, some of Greg Chaitin's easier lectures – all works with great epistemological interest – are practically, as a publisher might advertise, 'formula free'. And Bob Osserman's great expository – and not only – work, *The Poetry of the Universe* is also of this kind. But look at David Foster Wallace's *Everything and more: a compact history of infinity*. The amount of 'hard' mathematics in it makes it impossible to a general reader to read – and the reviews I read from mathematicians found his arguments totally unconvincing. (It helps if paramathematical writers know some mathematics!) It is a well-tested truism, that people with profound understanding of a field can usually speak about it in very simple, jargon-free language, and part of the importance of paramathematics has to do with this, the going for the wisdom rather than the knowledge, and the knowledge rather than the information. Of course (as in chess too – but less so) in mathematics often this is inextricable from the formal, combinatorial/calculational work. But not *always* – and paramathematics should tend to stay in that other dimension, as the works stated above (but one) brilliantly do.

So, to try and apply this to the online world: again, there is the difficulty of 'changing gears'. I find that players of online games, on the whole, like to operate at a certain level and do not like huge jarring changes. Thus, if a user is happily clicking away, he/she will not like to suddenly have to stop and read a twenty-page essay. I think this should be the main criterion online, not so much a narrowness of levels, as a new criterion of what types of thinking, whether narrative or lower-level, can be given a more or less homogeneous hypertextual style.

Christine Suurtamm

Would it be possible to teach secondary mathematics in ways that secondary English classes are taught? Students in English classes read great literary works, examine them in detail, look for themes, problems, dilemmas and also work at writing their own stories, poems, and essays. In mathematics, students could read and study great works of mathematics, examine some of the mathematical dilemmas that have been faced and resolved, and also be working at creating their own mathematics proofs and solutions. Would this create a more meaningful program of study?

I am all for the underlying idea in this, were it not for one problem: the texts themselves. The older ones (Euclid, Archimedes, Pascal, Descartes and Newton, say) which might be more approachable mathematically would be extremely tedious, if not impossible to read due to the antiquated notation. And the closer to us we come in time, the mathematics becomes too complex to be understandable in the original texts. But on the other hand, it is ridiculous: a student of English has come in contact with Shakespeare, Dickens, Emily Dickinson, Melville, Eliot, Joyce and whatnot, and a student of mathematics thinks humanity's achievement in this field is second degree equations and trigonometry, the equivalents of 'Mary had a little lamb', in a way. And this is a shame – no wonder students don't take mathematics seriously and are not fascinated by it. So, the challenge here is to create the right text: books introducing mathematical history but with the paramathematical, sophisticated slant, for various age groups. To me, there would be no more ambitious project in mathematics education work, and no better spent money by a well-meaning foundation, than to invest in the creation of such texts. I'd love – and may eventually get to – work on one of them myself, though I cannot guarantee the results of course.

Is proof really a story? Or is the evolution of the proof the story?

To speak somewhat schematically: the proof is the 'bare bones', the x-ray of the story. Yes, the proof itself is in a sense the story, but it is too dense to be narratively interesting, and too clean cut too. Obviously, as such, it has no great human interest. To acquire such, two things must happen: a) the human stories of the creators (I don't mean their love-affairs, I mean their dramas around finding the proof) must enter the scene and, b) the proof must also be seen as an epistemological advance, in context, and also partly through its metaphoric significance, in areas of knowledge other than that in which it strictly appears. (The metaphors can also be endo-mathematical of course, too, generalizing results to other fields, or extracting the strategic methodology.)

What makes a good mathematical story? Is the mathematics enough?

The prevailing wisdom in the world of fiction is that you can make a good story out of anything. And I think in some sense the same would be true in mathematics, as regards the underlying logic. But of course, the added, human dimensions (the story of the discovery, the importance, other important factors in the historical, epistemological, psychological or social sense) count much more for its rhetorical potential.