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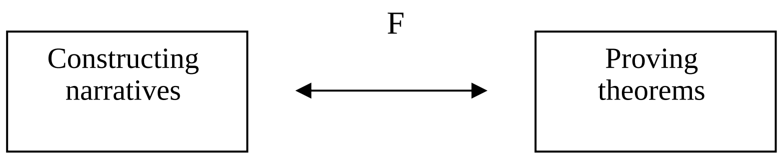
Euclid's Poetics

An examination of the similarity between narrative and proof[1]

1. Introduction

I want to state and briefly explore what I believe to be strong structural analogies between making narratives and proving mathematical theorems – analogies a mathematician might be tempted to call 'isomorphisms', i.e. one-to-one correspondences of the elements of two sets that, additionally, preserve their structure. My thesis does not lay claim to the rigor of a purely mathematical result. But I hope that by being even approximately accurate, it points in an interesting direction.

In graphic notation, I want to show that:



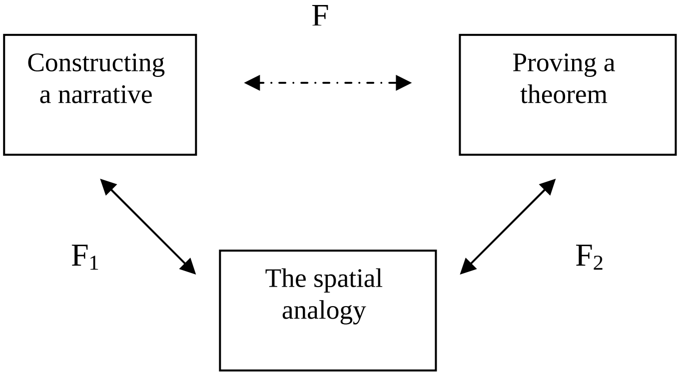
The idea of this isomorphism, which I'll call F, has been on my mind for quite a while, but really began to solidify (the case of a drop making the glass overflow, really) when I heard a reader of my novel Uncle Petros and Goldbach's Conjecture, comment that the story it tells "unfolds much as solving a mathematical problem."

Pursuing this analogy, I would like to give arguments for my main thesis much as 'solving a mathematical problem'. The technique I will use will be an application of the transitive quality: to prove that A is equal (or isomorphic) to C, it is enough to prove that both are independently equal (or isomorphic) to a certain B.

Thus, A=B & B=C implies that A=C

[1] Lecture given at the Mathematics and Culture conference, Venice, April 2001.

This common reference point is in the case of my thesis a spatial analogy, which I believe underlies both narrative and proof.



Thus, I will try and show the structural equivalence, F, between constructing a narrative and proving a theorem by showing how both of these are independently equivalent to a spatial model, the equivalences F1 and F2. The transitive quality then guarantees that F = F1 & F2

2. The underlying spatial metaphor of narrative

Since Aristotle's *Poetics* there has been an attempt to find universal laws underlying the structure of narrative. Interestingly, the most important insights were achieved in the twentieth century by theorists operating outside the field of literary studies proper. The Russian folklorist Vladimir Propp, in a seminal essay finds that the so-called 'magical folktale' always conforms to a particular structure involving standard 'functions' (his term) that can range over a set of variables, giving different versions of a more or less constant underlying structure. Roughly, this structure is:

a. The hero lives in a condition of stability.

b. Something upsets this condition.

c. The hero embarks on a journey to restore stability.

d. He faces challenges assisted by a 'magical assistant', who is often an animal.

e. The final challenge(-s) are successfully faced.

f. The hero comes to a higher state of stability, because of his actions.

What is important to my thesis is that underlying all these phases there is a journey to (geographical) points of which every phase of the journey can be associated: crucial encounters, acquisition of information or objects, challenges, fights, magical events, revelations, etc., all can be laid out, as it were, on a map, every step of the hero having a spatial analogue. These are often charged (but don't need to be) with a metaphorical resonance. Thus, advancement of the story is forward movement, decisions are cross-roads, the narrative goal is also a physical destination, etc. and there is of course the full process of coming full circle, from stability, to instability, to stability.

Anthropologists and historians of religion later generalized this kind of narrative structure, speaking of the 'quest of the hero' as the archetypal myth, a thesis presented in Joseph Campbell's famous book, *The hero with a thousand faces*. In more recent years, help has also come from the unlikeliest place: Hollywood. Trying to codify the underlying structure of a film-script, scriptwriting teachers and 'script-doctors' (sic!) have resorted to Propp and Campbell, seeing in the pattern of the quest myth almost universal validity, as the sort of *Ur*-story, the primal, archetypal narrative. And although their insights have resulted, largely, in an endless torrent of highly similar and very often vacuous films, their analysis has a lot going for it. By looking at countless stories, whether they be recorded on film, the page, or retold by the human voice, one can see that most of them conform essentially to this pattern: **a hero wants something and embarks on an adventure-laden journey to get it.** This 'something' that the hero wants (be it a person, an idea, a material object, whatever) is the goal of the journey or, speaking spatially, its destination. If we further generalize the definition of the quest myth and replace the '**hero wants something'** with the '**hero wants something *or* the author wants something for him/her**', then this encompasses practically all narratives or, to be exact, practically all *simple* or *elementary* narratives, as often a longer narrative, say a novel by Dickens, is made up of a combination of many simpler ones.

Let us look at some famous examples of heroic goals/destinations:

HERO	GOAL
Ulysses	Ithaca
Oedipus	Cure of the plague
Lancelot	Guinevere, the Grail
Hamlet	To revenge father
Romeo	Juliet

Juliet	Romeo
Jay Gatsby	Daisy
The three sisters (Chekhov)	Moscow
The old man (Hemingway)	The fish

Now, the hero's journey may be very literal (as, say, in the *Odyssey*) or very metaphorical (as in T. S. Eliot's *Four Quartets*) and is often both at the same time, as for example in the medieval legend of the Grail. But whether metaphorical or literal or both, what's essential to our discussion is, again, that the hero's journey can be *mapped* (interestingly, a geographical as well as mathematical expression) i.e. can be given precise spatial form, even if this 'space' can also be immaterial, as is, say, the world of memory or imagination.

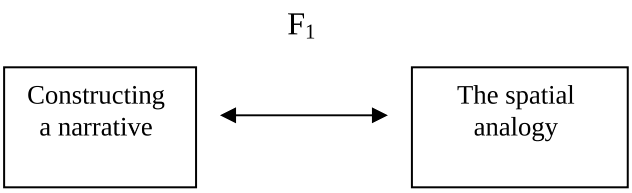
As to the hero reaching the destination, literature has gone a long way beyond the alternatives of the traditional quest myth, a Gilgamesh, Odysseus or Parcifal, and their various versions of a 'happy end'. The reaching of the goal (destination) can take many different forms, as for example[2]:

1. The goal is reached and this fulfills the hero's need.
2. The goal is reached but the hero finds he is disappointed with it.
3. The goal is reached but then the hero realizes a new goal lies ahead and thus embarks on a new journey.
4. The goal is reached but this only makes the hero realize the importance of the journey over the goal.
5. The goal is only partially reached and the hero realizes and accepts this.
6. The goal is only partially reached, the hero realizes and does not accept this.
7. The goal is not reached, and this makes the hero sad.
8. The goal is not reached but that's alright, because the hero has reached a new insight.

[2] Readers may amuse themselves by finding cases illustrating each case, from literature or the cinema.

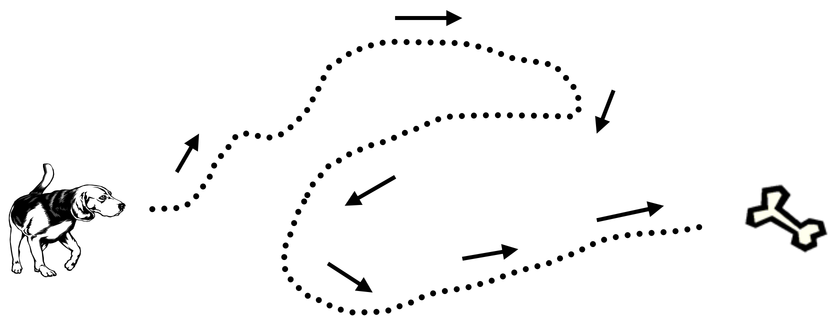
And so on. To summarize: **almost all stories have to do with a hero wanting to (or the author wanting the hero to) get something**. This can almost always be translated, structurally, to the wanting to get *somewhere*, by following a certain course, literal or metaphorical. Thus, any narrative can be represented as a journey, with a beginning (B) and an end (E) with various forces (arrows) operating as either ‘helpers’ (Propp’s term) external or internal, or obstacles, influencing the course of the hero’s progress. Dotted lines here indicate ‘the roads not taken’, in T. S. Eliot’s famous phrase, i.e. alternative courses the hero did not finally choose.

This more or less settles the first part of our argument, i.e. that there exists an isomorphism, which we called F_1, between narrative and a spatial model.



3. The underlying spatial metaphor of mathematical proof

I first hit upon the idea of the spatial analogy also underlying mathematical proof when reading in the *Homilies on the Hexaemeron* of the fourth century Christian theologian Saint Basil, his wonderful insight that the dog (yes, the *dog*) can be credited with the invention of the mathematical method of *reductio ad absurdum*.



You see, when a dog searches for the desired object (bone) he will begin to sniff a likely trail and if disappointed will retrace his steps somewhat and start off in a new direction. Obviously, this brought to Saint Basil's mind the method of someone like Euclid, when saying: 'let us assume that the primes are finite, and see what happens'. (As is well known, Euclid then follows the consequences of this hypothesis and since it brings him to a contradiction applies the principle of the excluded middle to conclude, in more modern terminology, that if 'not P is false, then P must be true'; or, in our example, that since the primes cannot be finite, they must be infinite.) But this too can be expressed with a simple algorithm, which is really spatial: 'at a crossroads, forking into roads A and B of which one leads to a cul-de-sac and the other to the treasure, first take A. If it leads to a cul-de-sac, then retrace your steps, take B and be led to the treasure with certainty.'

Here, we must make the crucial distinction between the proof of a mathematical theorem as it is experienced by a student/reader studying an already discovered, published result, and as it is was originally established by the mathematician(-s) who discovered it. It is this second viewpoint that is more interesting, although of course a published proof may contain, in an often indirect sense, part of the intellectual adventure of its completion. The process of proof can be very simple (again, see Euclid's proof of the infinity of primes) but it can also be long, arduous, complicated and multi-faceted. A good example of this is Andrew Wiles famous proof of Fermat’s Last Theorem, which was the culmination of a very long process lasting a few decades (or centuries, if you want to go back to Galois and the origins of modern algebra) and was successively created (although with no clear end in sight, for a long while) by a number of mathematicians, among them Taniyama, Shimura, Weil, Frey, Ribet, and a few more with Wiles providing the final integrative thrust that brought the various threads together.

Like a narrative, such a process of gradual discovery, whether long or short, complex or simple, can be mapped, i.e. it can be given a spatial form. In fact, more or less everything we said comparing the narrative to the spatial model holds also true of the process of mathematical proof.

Let us investigate this point: a mathematician starts out wanting to prove a proposition, which is really the end of his *destination*. (Of course, he may also start out, like a hero in some modernist fiction, merely by fooling around with ideas, with no destination, just a general sense of ennui leading to curiosity, leading to questions).

Here are some examples:

THE HERO	THE GOAL
Euclid	The primes are infinite
Newton/Leibniz	How to find gradient of curves
Evariste Galois	The solution of 5th degree equations
Henri Poincaré	The Three Body Problem
Atle Selberg	Elementary proof of the Prime Number Theorem
Stephen Smale	The higher-dimensional Poincaré Conjecture
Andrew Wiles	xn + yn = zn admits no integer solutions for n>2

Most aspects of the process of proof will admit a spatial correlative:

- The mathematician moves forward (often backwards, on sideways) in logical space, searching this way and that.
- The mathematician may take advantage of road maps, of greater (already proven results) or lesser (conjectures) accuracy.

- The mathematician will face challenges, disappointments, will win some fights (intermediate results) and lose some (cul-de-sacs), may often change direction, will be assisted by ‘magical assistants’ (mentors, colleagues, the accumulated knowledge of the past), may employ powerful talismans or weapons (new methods) and will finally (in a ‘happy end’ scenario) reach his destination – i.e. prove the desired theorem. All these have their analogues in logical space, which we can envision as a decision-studded magical, metaphorical forest.

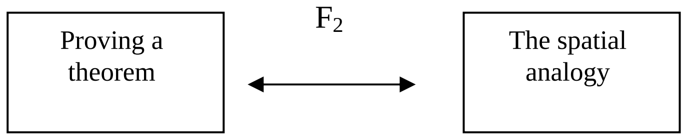
Of course, the happy ending is not obligatory. The mathematician may not reach his goal, or find it not at all similar to his expectations (Nagata working on Hilbert’s Fourteenth Problem only to finally prove it false), or, again like some modernist hero, may think that he has arrived, while he really hasn’t – like Fermat thinking he had proved his theorem when (we think now) he hadn’t.

In fact, the possible outcomes of his spatial progress into the forest (maze, labyrinth, whatever) may end in some of the various ways that we thought were reserved for fiction. Thus, for example – and I am here taking similar options to those presented earlier (for ‘mathematician’, read ‘hero’):

1. The goal is reached and this fulfills the mathematician’s need (e.g. Euclid and the infinity of primes.)
2. The goal is reached but the mathematician and/or others are disappointed with it (e.g., or the proof of the famous Four Color Theorem, which was so cumbersome that some do not accept is a proof.)
3. The goal is reached but then the mathematician realizes a new goal lies ahead and thus embarks on a new journey (the proof of a theorem points to a much more important result).
4. The goal is not reached but this only makes the mathematician realize the importance of the journey over the goal (while trying to study the distribution of primes, Riemann invents his zeta function.)
5. The goal is only partially reached and the mathematician realizes and accepts this (proofs that don’t manage the full result but a weaker version of it, e.g. Jing-Run Chen’s proof that every even number is the sum of a prime and an almost prime – a weaker version of Goldbach’s Conjecture).

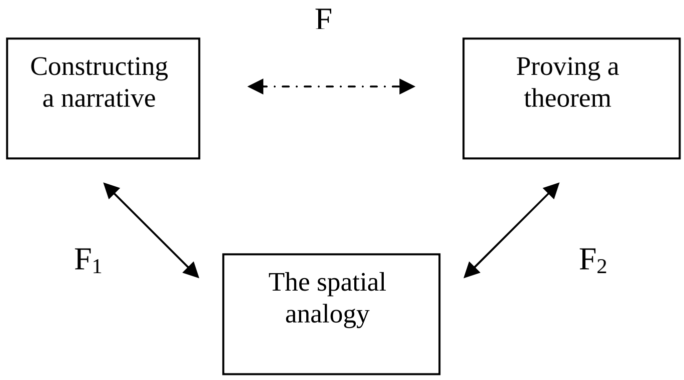
6. The goal is not reached but that's alright, because the mathematician has reached a new insight (Galois failing to find a formula to solve the quintic equation, but discovering group theory and a lot more on the way).
7. The goal is only partially reached, the mathematician realizes and does not accept this. (Alas, countless examples.)
8. The goal is not reached, and this makes the hero sad. (The same.)

These arguments seem to take care of the second part of our proof (oh, call it 'argument', if 'proof' sounds too strong), demonstrating the isomorphism:



4. Conclusion

We now seem to have come to the desired point, completing our argument with the isomorphism F, between narratives and proofs, by virtue of the transitive quality (F_1&F_2 implies F):



I suspect that this analogy (isomorphism) does not seem too interesting to a mathematician or, not to be unfair to the more poetically inclined, it doesn't seem *useful*. Knowing that proving a theorem may look somewhat like the unfolding of a story is perhaps no help to a mathematician in proving new theorems – and, like it all not, this is the prime criterion of usefulness to mathematicians. But the analogy may be more useful to people dealing with narratives. Although it will not answer

Hollywood’s dream of a magical formula to create more interesting narratives, it does point at a handy formalism, and at analogies that can provoke a storyteller’s thoughts.

And what more can a storyteller want?