Apostolos Doxiadis

Embedding mathematics in the soul: narrative as a force in mathematics education

Opening address to the Third Mediterranean Conference of Mathematics Education
Athens, January 3, 2003

Ladies and Gentlemen,

I feel as I stand before you today slightly like the repentant sinner we see in films of the American South, describing the story of his salvation to a congregation of believers. I was a hater of mathematics once. In fact, I was a fanatical hater, until age fourteen – and this through no fault of my own. It was my teachers who were the culprits. As also, it was a teacher who suddenly, miraculously, made me a passionate lover. His way – though I'm not sure he did it consciously –, was the stories he told me stories of mathematics. They were few, but enough to set the ball in motion. This led me to other stories I read – and then I was hooked.

In a recently published little book, in which Field Medalist Tim Gowers attempts to explain to the *hoi polloi*, the un-mathematical multitudes, the beauties of mathematics, the dedication to his wife reads: "To Emily, in the hope that it will give her a small idea of what it is I do all day". To us, who know something about mathematics, this rings a bell, on a melancholy note: Mr. Gowers is far from alone in her ignorance. Alas, mathematicians live and work in a *terra* that is *incognita* to the population at large, i.e. to everybody else.

This is a particularly apt point to begin my lecture. I believe with many of you that one of the most basic, if not the most basic, problem in mathematics education is the generalized dislike of mathematics in our culture. Whether we like it or not, it is a fact: a great number of people dislike mathematics. And I use the word 'dislike' to cover a full spectrum ranging from the outright 'haters' – as I used to be before I saw the light –, going through a scale of negative feelings, from malignant apathy to aversion to mere indifference – and indifference is often the worst of the lot.

The infra-structure of this dislike is complex. But at the root of it are abstraction and irrelevance to life as lived. More than anything else, it is these two monsters, that block the gate.

In what follows, I will speak from my own viewpoint – the viewpoint of a storyteller. But please remember this: the first, though not the last, aim of storytellers – and in this they differ dramatically from mathematicians – is to please the audience. I know: "pleasing" and "audience" are not words frequently used in mathematical circles. But to me, a storyteller, the students of mathematics are an audience. And by the end of this lecture, I hope to have imparted some of my attitude to you too.

As for pleasure of pleasure, you certainly know about its role, as theatre- and film-goers, as readers of novels, listeners of music. To derive from a book, a play, a film, a concert, a higher 'message' is highly commendable. But unless you are also entertained, there is no hope of instruction. If you close the book in boredom after ten pages, or take a little siesta halfway into the film, you will definitely not receive the message, however 'high'.

Ladies and Gentlemen.

If people were logical machines & education the equivalent of programming them, like computers, what I have to say would be irrelevant. Yet, as every teacher can tell us, these hypotheses don't apply. On the contrary, education is – should be, at its best – a process involving the complete human being. And as the wise always knew, and the philosopher – and also, incidentally, ex-mathematician – Edmund Husserl told us, human beings do not perceive passively, but with intention. We see more when we want to see more, we learn more when we are eager to learn more. Our cognitive processes are guided by our motives, our reasons for wishing to make sense of this world.

Which is perhaps a complex way of restating the old proverb that 'you can take a horse to the fountain but you can't make him drink'. So: my concern won't be the contents of the rich fountain of mathematics education – it is your job to explore it, and you shall do so worthily, to judge from the program. My role is to see how we get the horse to the fountain and – particularly this – create the desire to drink.

But, think of this: how unlikely my presence at your conference would be a decade ago. What makes it possible that I, a storyteller – and, only incidentally, an ex-mathematician – address you today, is that in the past few years a benign revolution has begun: amazingly, mathematics has suddenly become the subject of storytelling!!! …Like all good revolutions, this one too was started by people outside the establishment, the pariahs, the ex-communicants, and the imported labor. My appearing before you is due to the fact that I too, in my humble and somewhat angry way, am one of the rebels.

Of course, I am referring to the unexpected, miraculous one might say, 'coming of mathematics center-stage', to use an expression of professor Robert Osserman. A book recounting the story of the solution of an old, highly abstract problem (*Fermat's Last Theorem*) becomes a bestseller. So does the biography of a great mathematician (Paul Erdös) and then another (John Nash) – this one also becoming an Oscar-winning film. And, at the same time, invented narratives appear, narratives with mathematicians as their heroes and mathematics – if not their subject – at least their setting. A successful film has the unlikely name of 'π' – not as in 'apple-pie' which would be understandable, but as in the ratio of the circumference to the diameter of a circle. A play by the name of *Proof* – yes, mathematical proof – wins the Pulitzer Prize… And novels including the words 'theorem' or 'conjecture' in their titles are sold internationally. And what's even stranger: they are read.

A small personal anecdote will illustrate further: a few years ago, a publisher of my book *Uncle Petros and Goldbach's Conjecture* asked me if I knew the title of my next project. I said "yes, it is *Aunt Caterina and the Riemann Hypothesis*." I was joking of course, but I realized, shocked, that the publisher wasn't: I had made him happy. The fact that he had no clue of what it was that Riemann hypothesized, or who Riemann was, did not decrease his delight. Oh, this must be a miracle, after all, I thought: mathematics is suddenly a hot-seller!

You know it better than I that mathematics education does not occur in a vacuum. And I am not referring here to the societal or psychological realities which also set their own requirements. I mean, more specifically the endo-mathematical influences: what occurs in mathematics education is to a very large degree a reflection of the prevalent self-image of mathematics. It is what mathematicians think of their field and of their role in it, which provides the underlying paradigm for mathematics education.

To locate the time of the split of mathematics from the rest of the culture, and with it of the birth of the dislike of mathematics we have to travel a longer distance. But be comforted: the distance is long only in time. Not in space.

And here are the principal culprits:

They are the three gentlemen circled (by me, not the Thought-Police) interestingly occupying the apexes of an isosceles triangle in Raphaello's sublime 'The School of Athens'. And if Raphael got his geography mixed-up, and Pythagoras (lower left) taught in Southern Italy, while the old fox, Euclid (lower right, demonstrating a proof) in Alexandria, the spirit of this a-historical assemblage is correct. For it was Plato (top center, pontificating) and his Athenian school that encapsulates the essence of the problem.

But I beg you: like good Pythagoreans, let's keep what I said to ourselves. Don't tell anyone. The last thing we need is to hear on the evening news that some barbarian blasphemed against our glorious past and to have our conference picketed tomorrow by some fanatical group defending it. But just to make sure, I'll say it clearly, though of course it's too obvious to need my opinion: The Greek achievement was colossal, it was revolutionary, it is the genes out of which following generations built the splendid edifice of mathematics. But those genes also contain the cause of the problem.

Yes. Mathematics as we know it is, to a large extent a Greek dream come true.

But in this dream, lies its *esotericism*, the sense of an ivory tower, an insular domain of higher, knowledge: of the close-guarded nature of the Pythagorean brotherhood, to whose members only mathematical truths could be imparted, think of the legendary *Αγεωμέτρητος μη εισίτω*, the notorious 'no-one ignorant of geometry shall enter', written over the gate of Plato's academy – now interestingly the motto of the American Mathematical Society. In this dream also lies abstraction, the abstraction that makes most of mathematics possible: think of Socrates idea that triangles exist somewhere out there in conceptual Heaven. And think of the child of abstraction, the exalted irrelevance, the 'uselessness', which G. H. Hardy so proudly defends in his *Apology*. Remember the story of Euclid telling haughtily to his slave to give five drachmas to the student who dared ask what practical benefit he would have by learning a certain theorem. Remember Archimedes, the prototype of the absent-minded mathematician, caring only of his circles as a Roman soldier is about to slaughter him. And, of course, in the heart of this Greek dream lies formalism – not as a philosophy but as a way of conceiving mathematics – and foundationism (these are principally Euclid's work), the concept that mathematics is built from the bottom up, and proceeds in clear, rigorous, verifiable steps, upwards, upwards, upwards, a Tower of Babel of absolute knowledge whose illusionary nature was first exposed only in the 1930's by a young Viennese graduate student, by the name of Kurt Gödel.

Of course: abstraction, irrelevance, purity, formalism, make for good mathematics – though I shall have to say something about this too in a while.

But, sadly, they make for bad mathematics education. Each one of these concepts – abstraction, irrelevance, purity, formalism – pushing mathematics further away from a growing human being, a being whose psyche is in the phase of its development that – no soft-brained psychologist but a great mathematician, Alfred North Whitehead –, calls the Romantic Phase.

Yet, after more than two millennia, the Platonic-formalist *Weltanschauung* still prevails. And to a large extent mathematics education is a Procrustean struggle to enforce the direct Mathematics-to-Mathematics-Education homomorphism into the mind of the cognitively, psychologically, intellectually and spiritually unprepared student.

A while ago, I called the invasion of storytelling into mathematics a revolution. The reason for this is because I do believe that our recent mathematical narratives, especially those that significantly include mathematical or, as I shall call it, paramathematical argument in their plot, transcend mere exposition or, that terrible world, 'popularization'. On the contrary, I believe that they point to a new and original way to look at mathematics itself. This is a crucial point to which I shall return, as it sets the basis for a non-formalist, non-Platonist a view, a view of mathematics not as something pinned like a dead moth for Euclidean purists to examine – and in this form taught to our unprepared children –, but mathematics as it is lived by human beings, as it is loved, as it is explored, feared, created, dreamed of... By human beings. This new mathematical storytelling is inspired – in my view – by a rebel spirit. And I believe it is about time that this rebel spirit begin to infest mathematics education. I shall clarify as I proceed…

I am sure many of you know the work of developmental psychologist Jerome Bruner – his experiments, somewhat in the line of Jean Piaget, are the groundwork of much in modern educational theories. In a less well-known work of his, the essay 'Two Modes of Thought', Bruner makes a crucial point: what we call thinking can occur in the human mind in two distinct modes, modes that are irreducible to one another. The first, which he calls *paradigmatic*, is the logico-deductive, a mode that uses as its tool logic & aims at classification. The second mode Bruner calls *narrative*. Everyone knows stories are important for human life. The importance of Bruner comes from his calling storytelling, specifically, a 'mode of *thinking'*, bringing also to my mind the wise art historian Ananda Coomaraswamy, who caught my heart when he said that 'art is a form of knowledge'.

It is not difficult to accept that in the human sciences, narrative can actually *be* thinking.

Top-grade historical, legal and judicial narrative discourse is known from antiquity – and if you have any doubt this is *thinking*, then read Thucydides or Cicero. (Incidentally, it's about time someone told our history teachers to stop torturing ten year-olds with the 'causes of the Persian wars' and open their hearts and minds with the stories themselves, told by that sublime storyteller, Herodotus.) Then medicine joins in – the great invention of the case-history, by Hippocrates – and in modern times psychology, anthropology, politics, and more fields use narrative as the *prima* *materia* of analysis. In these fields narrative thinking dominates. And this thinking is irreducible – on pain of meaninglessness – to the classificatory mode.

But in mathematics? Mathematics so obviously fits Bruner's first mode – in fact, it is the very prototype of a logico-deductive science – that it's hard to see its essence as anything but classificatory. In fact, see how often the main task of a mathematical field is a full classification of its objects of inquiry, be they finite groups, knots, Abelian varieties and whatnot. Classify them all, neatly fit every object in an abstract, non-linear space of relationships (what mathematicians would call a non-linear graph) and you are done.

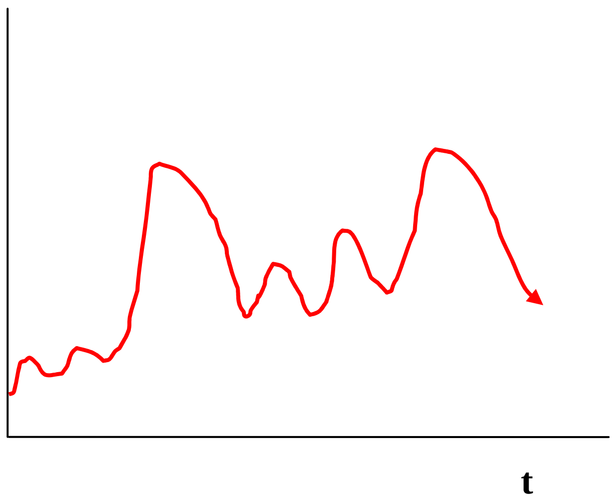
Of course we shouldn't be hard on this non-linearity: it is the basis of good science. It allows the mind, with its extension of writing, the page, the pen, the book, the computer, to conceive and to fix patterns that are almost impossible to construct or even visualize in life-as-lived.

But again, non-linearity is bad communication and thus, of necessity, bad teaching – especially at the younger ages. The reasons for this have to do with the way the human mind is constructed as also with the way human life is lived. For better or worse, human beings live in time. And time is linear. And…

…Only in time can the moment in the rose-garden,
The moment in the arbour where the rain beat,
The moment in the draughty church at smokefall
Be remembered…

As T. S. Eliot tells us, in 'Burnt Norton': *Only through time, time is conquered.*

Let's see what happens in storytelling:

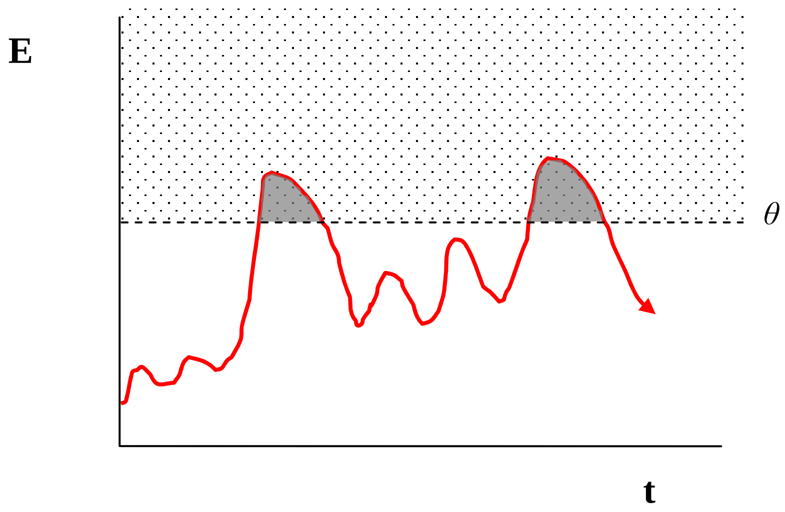


A story (red line) starts at a point 0 and progresses, in time, to its end. That is the one dimension that is essential, event following event following event. John loves Mary. John marries Mary. But then Mary loves Jim. So John kills Jim – or Mary, if he is a chauvinist.

But, as you can see, it is the value on the y-axis, that causes the fluctuations As in human life, the dimensions behind a story are limitless, the time-line can curve and twist in an infinite variable Hilbert space. But what makes a story appealing and understandable to a human being, is Emotion. And 'understanding' here means something different than in mathematics. Oh, I am not madly in love with the Pythagorean theorem, but I understand it well enough. But I am very much moved – I *understand*, you see – *Romeo and Juliet*, because I like it; in fact, because I like Juliet.

Stories – like all art – speak to us only to the extent that we are touched by them. A fact that storytellers and story-scholars and, alas, yes, Hollywood producers too, know all too well. Its prerequisite is something called identification.

By some weird alchemy, the necessary condition for the reader or viewer of a story to be moved, is that its heroes are also moved, positively or negatively. And identifying with the heroes, through empathy, we share their passions.

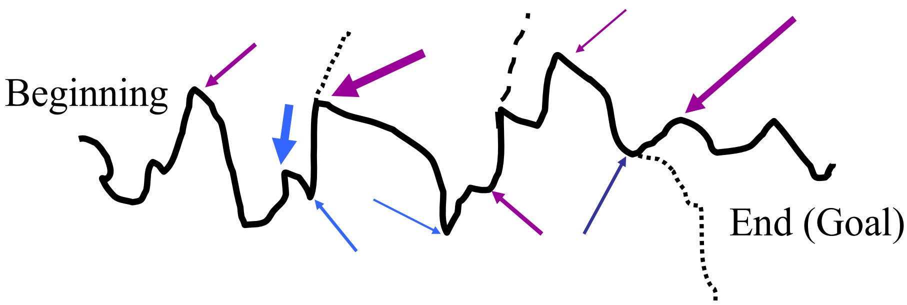


In this diagram, we see that identification is achieved if the hero's or the audience's emotions (let's equate them, for simplicity) go above a certain threshold level θ – obviously θ is a variable of both story and individual receiver.

Whether we like it or not, identification is a prerequisite for the understanding of a story. I know of course that to mathematically-trained minds, like yours, the emotions of the audience are shaky ground on which to base an epistemology. But remember this: only in its more trivial cases, like love stories of the *Harlequin* variety, or mass-produced television material, is pleasure the sole aim of storytelling. Good stories create bumpy emotional rides in order to communicate underlying patterns. Pleasure, identification, is the medium, not the message.

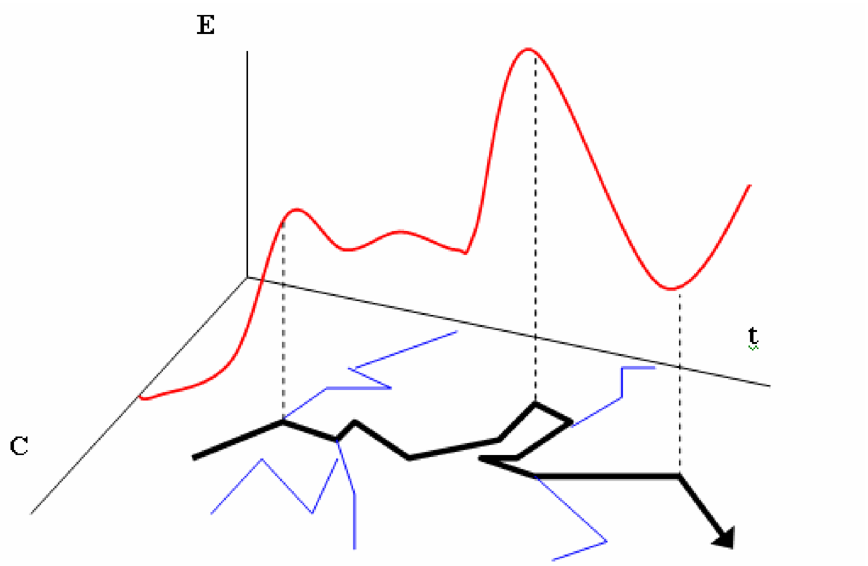
In fact, a good story is linear only in appearance, its temporal shape really a reflection of inner, complex structure.

To begin with, every story is a journey:



Where the hero advances from the beginning to the end – the space here is conceptual –, meeting enemies/obstacles/negative situations (purple arrows) and friends/helpers/positive situations (blue arrows). The dotted lines represent some of the – to refer again to T. S. Eliot – roads not taken.

Thus, the red line that we saw earlier, is nothing but a projection of an underlying reality.



I this diagram, C represents Choice. We see now that the red line, the emotional line, is but a reflection of a progress, re-enforcing its meaning. Again, emotion is the medium: the message meaning is a complex reality of decisions – something that seems much friendlier to reason.

Quiz: if we conduct a survey, asking persons in the street to find the name of a great twentieth century mathematician among the following: 1) David Hilbert, 2) Herman Weyl, 3) John Von Neumann, 4) Alexandre Grothendieck, 5) John Nash… What do you think the answer will be? Stupid question: John Nash, of course! But is Nash a greater mathematician than the others? No. Why then? Another stupid question. And what about Andrew Wiles? The fact that he is quite well-known outside the mathematical community – a unique feat for a mathematician – is not due to the fact that he proved Fermat's Last Theorem (to the extent that he did: he really proved a much stronger, but much less famous, hypothesis – it was Ken Ribet who had proven, before him, that this also implied FLT). No. Wiles' fame is really due to the fact that someone wrote a nice book about it.

If you find this state of affairs unfair – as it is, in a way –, I offer as consolation the opinion of psychoanalyst Bruno Bettelheim, from his book *The Uses of Enchantment, the meaning and importance of fairy tales*. When a fairy tale instructs and delights a child, he tells us, it does not delight because it instructs. On the contrary: it will instruct only because – if and only if, I should say – it will delight. If *Jack and the Beanstalk* manages to teach a child something about the rights of the weak and the value of courage and persistence, it is not because the rights of the weak, courage and persistence are in themselves appealing to a child. This is crucial: the child will learn those good things from the fairy tale only because he *likes Jack* and thus identifies with him. If the Giant, rather than Jack, happened to be the more likable character in this tale, then it would teach the child another lesson: that all intruders into the houses of the privileged must be punished harshly, preferably cooked and eaten in a stew.

So: if the new surge of appealing mathematical stories motivates some people to learn more about mathematics – and it does – this is a welcome consequence. But they have eaten the pill for the sugar, not the medicine.

And if I am allowed to speak from personal experience: almost daily I hear people tell me that their view of mathematics was changed – for the better, fortunately – after reading *Uncle Petros and Goldbach's Conjecture*. And I get the full range, from the lovers who feel I have added to their passion, to the dis-likers who now have turned to liking, even cases – and those are most dear to me, because they remind me of my own past –, of haters-turned-to-lovers. Such is the power of narrative. In fact, this power is so great that on occasion the criterion of 'good story' takes completely over, as in the invitation I received two weeks ago, by a provincial branch of the Hellenic Mathematical Society, no less, asking me to, quote, "give a lecture on the life and work of the great Greek mathematician Petros Papachristos". It doesn't matter that Petros is a fictional creation. If you love him, you want to find out more about his work.

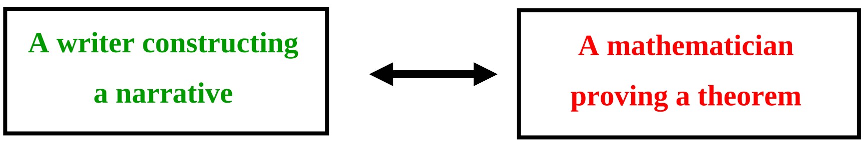
So: we obviously have at least one valid reason – and it is a very important reason – to introduce narrative to mathematics education: to try and move people some distance on the scale, from hate towards love.

Yet, there is more, a deeper and more intricate connection between narrative and mathematics.

Aristotle says that drama is about *character revealed through action*. But the action of mathematical heroes is mathematical – and that deserves our extra attention.

I will briefly outline an idea I published recently in brief form. I call it "Euclid's Poetics".

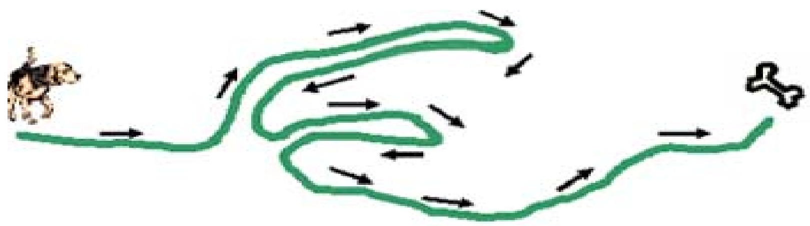
This is the essence of it: that there is a strong isomorphism between the underlying structure of story – any story – and the process of proving a theorem. In fact, the paradigm of mathematical exploration is the purest language in which to express story construction.



I got the first hint of the idea reading a comment by a reader of *Uncle Petros* on the internet, that "the book unfolds much as a solving a mathematical problem would". Yet, reflecting on this, I realized – shocked – that all books, all stories that is, unfold much the same way.

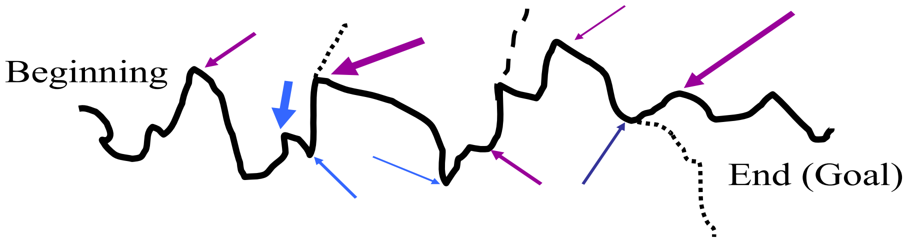
And this brought to my mind something I'd read many years back in the unlikeliest of places, a theological treatise by Basil of Cappadocia, that same person, the *Aghios Vassilis* whose carols we heard sung in Athenian streets a few days ago. In this treatise, the *Homilies on the Six Days of Creation*, Saint Basil propounds the thesis that God created the world with an anthropocentric goal in mind. Thus, the Sun is in the sky to light us on our way to work and worship, the sea so we can travel and fish, and so on. When he comes to explaining the *raison d' être* of the animals – cows for their milk, bears for their skin, the horse to pull the cart, etc. –, Saint Basil says something totally amazing about the dog. The dog, you see, was not put on Earth simply to guard our sheep and our houses, but… But to teach us the *reductio ad absurdum*!!! Saint Basil uses the precise words, he writes: *εις άτοπον απαγωγή*.

Let me explain his point – the diagram is my own:



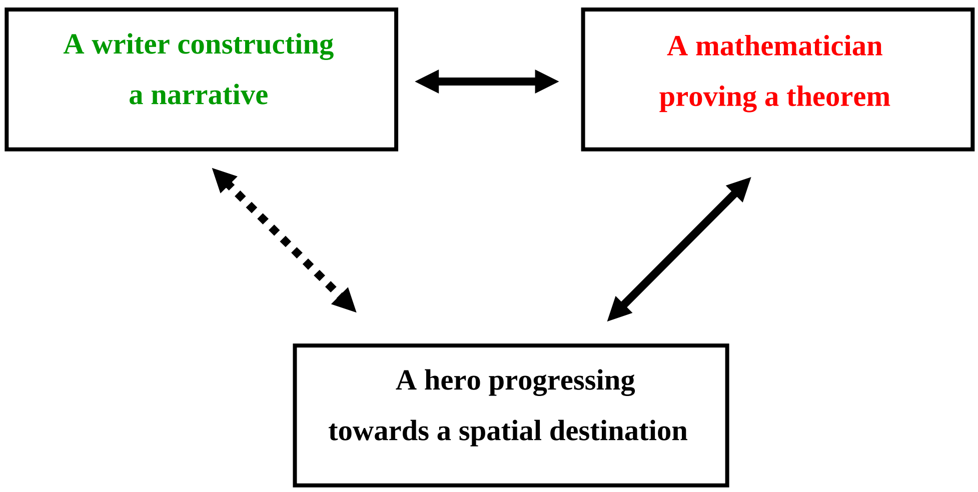
The dog, having smelled the bone will first explore one possible path. But if the bone is not that way he will retrace his steps, i.e. he will not automatically correct his trajectory – if he did that he would have taught us approximation –, but go back to the beginning, modify his premise – his course – and try again! …This particular dog in the diagram has never heard of the Rule of the Excluded Middle, so he has to do it twice.

The point here is extremely important: there exists an analogy between mathematical proof and spatial exploration. You see, the progress of a mathematician to the proof of a theorem can be mapped in some conceptual space, thus:



Where the mathematician advances from the hypothesis to theorem meeting enemies/obstacles/negative situations (purple arrows) and friends/helpers/positive situations (blue arrows). Here, as in the earlier diagram we see the projection in two dimensions of a course in a multi-dimensional space, also conceptual – but sometimes, as in the case of Paul Erdös also spatial! –, dimensions being purely mathematical and others human: thus, on the negative side, both mathematical and human, we can have cul-de-sacs, impossibilities, contradictions, lack of necessary intermediate results, but also antagonism, fatigue, etc., whereas on the positive side we have lemmas, mathematical tools, associates, articles, results, flashes of insight…

This leads in fact to another isomorphism and so by application of the transitive quality we have a full triangle:



Storytellers can use this triad to generate and clarify the structure of their stories. But, more importantly to us, mathematicians, paramathematicians, educators, can use the tools it gives to give narrative form to mathematical argument.

I will give you a taste of this, with two examples.

First, remember that like mathematics, narrative is problem-generated and goal-oriented: the backbone of a story is formed by this hero wanting something.

Thus:

Ulysses	wants	Ithaca
Oedipus		Cure of the plague
Parsifal		the Holy Grail
Hamlet		Revenge father
The three sisters (Chekhov)		Moscow
The old man (Hemingway)		The fish
Romeo		Juliet
Juliet		Romeo

Likewise, in every mathematical quest the hero (mathematician) wants to get (solve or prove) something.

Hippasus	Prove irrationality of $\sqrt{2}$
Newton/Leibniz	Find gradient/area under curves
Evariste Galois	General solution of 5th degree equation
Atle Selberg	New proof of the Prime Number Theorem
Stephen Smale	Higher-dimensional Poincaré Conjecture
Andrew Wiles	Fermat's Last Theorem

A straightforwardly simple progress in a proof process – as in the first case, here – is not always the case. In storytelling it happens in what we call romance, the happy ending, the simple – in terms of a constant goal, from start to finish –, quest myth, like Homer's *Odyssey*, or *Lord of the Rings*. Otherwise, the search can take any form, within certain constraining rules, e.g. that a repetition of the same fruitless sub-itinerary, in a loop, is permissible only if the hero (or mathematician) has other things on his mind. I will show two isomorphic variants of complication *vis à vis* the intended goal, to give you an idea of the analogy:

STORYTELLING: The goal is reached (i.e. the task is achieved) but…

a. **This points the hero to a more interesting goal (the case of Kazantzakis' Odyssey, which begins with Odysseus reaching Ithaca – and departing again, wanting more).**

b. **This makes the hero realize the importance of the journey over the goal (the case of Cavafy's Ithaca)**

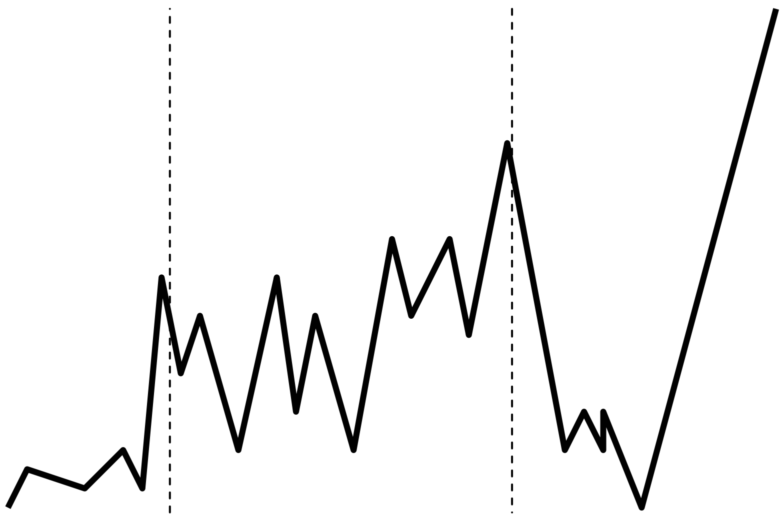
While, analogously, in mathematics:

MATHEMATICS: The goal is reached (i.e. the theorem is proved) but…

a. **This leads to a more interesting goal (e.g. Stephen Smale, solving a topological problem regarding the Horseshoe shape, and being led by this to non-linear dynamical systems and the invention of chaos theory)**

b. **This makes the mathematician realize the importance of the journey, i.e. intermediate results, over the goal (Evariste Galois, who, by showing the impossibility of the general solution of the equation of the 5th degree, created something far more important, group theory and a lot more).**

For the second example, here is a sketch from a Hollywood script-writing class – ah, Hollywood again:



1st ACT: Set-up	**2nd ACT: Conflict**	**3rd ACT: Resolution**
Beginning	**Middle**	**End**

This is a very classical three-act structure, again of the happy-end variety, an extremely effective machine, when handled well, for manipulating the audience's emotions, hopefully also to impart some meaning.

Now, see how well it applies to this most famous tale of mathematical proof, Andrew Wiles and Fermat's Last Theorem – in fact, it is exactly the perfect fit to such an archetypal audience-pleaser which explains the great appeal of the published form of the story of FLT.

a. First Act. Andrew Wiles, a successful mathematician, goes on with his life. Until (near the end of Act One) he goes to a friend's house to drink iced-tea one afternoon and there hears that Ken Ribet has proven the connection between FLT and the Taniyama-Shimura conjecture. Since proving FLT was his childhood dream (we are told) this immediately mobilizes him into action. This is what they call in Hollywood Plot Point One, a sudden peak in intention which engages the protagonist and spurs him on.

b. Second Act. This is the seven years lonely struggle with its ups and downs – don't forget that this is really an outline, showing mostly the emotions. By our previous diagrams there is a clear-cut mathematical progress underlying this, ending with Plot Point Two, the point where Wiles thinks he has proved the Conjecture (thus FLT, too) and announces this during a course of lectures, thus achieving instant world-acclaim – this, of course, is due to FLT's fame, and not its intrinsic value per se.

c. Act Three. After the peak at the end of the previous act, we have a huge drop when a serious gap is discovered – so serious in fact that Wiles confesses, in writing, that the theorem 'is not in fact proved'. But then, in the end, gloriously, just as he is ready to abandon his efforts, Wiles tells us, he suddenly sees the light, there comes the great, final insight that proves the Taniyama-Shimura conjecture and thus FLT too. Applause. Curtain.

Now, before I too move into the third act of this lecture, in order to make concrete proposals about the application of narrative to mathematics education, let me point to one possible paradox: I referred earlier to the existence of a strong homomorphism from mathematics to mathematics education. Yet, as narrative is at present extraneous to mathematics, does not my proposal of assimilating it into its teaching smack of a contradiction? How can a method of teaching mathematics have validity, if it does not reflect a part of mathematics itself? Here, I return to the concept of ‘revolution’, I referred to earlier.

Paul Erdös said, cutely, that mathematicians are machines turning coffee into theorems. Alas, there is more truth there than meets the eye.

This is precisely why I used such a strong term: ‘revolution’. I believe that a new branch of mathematics, not yet obviously apparent as such, not yet unified or with a constant voice, perhaps not yet daring assert its existence as an independent discipline, is being born. A new branch of which we have desperate need. I call it *paramathematics*. For mathematicians – of the ‘theorem proving variety’ – only speak about mathematics from within it, with its own criteria of rigor – considering and outside view trivial or irrelevant.

Yet it almost seems to be a direct corollary of Kurt Gödel’s great theorems, that mathematics does not possess the language within itself to reflect on its own nature – and how could it, when it cannot even (at the level of one theory) solve the problem of its own consistency.

This is the meaning of *paramathematics*; this is the need for it: for a field, literally, on the side of mathematics, as *metamathematics* is beyond it. This will of necessity be cross- and inter-disciplinary, drawing on mathematics itself but also on logic, philosophy, epistemology, the history of ideas, cognitive psychology, sociology, anthropology, education theory. And, of course, mathematics education.

Of paramathematics we require that it provide mathematics as we know it with context and thus meaning, extra-mathematical meaning and thus criteria, distance, clarity, bird’s eye view, integration with thought, history and society.

But it is obvious – good, handy mathematical expression this: “it is obvious” – that this field can develop neither along the strict logico-deductive model of mathematics itself (in that case it would be locked in the same kind of short-sightedness). Nor can it really adopt the criteria of a more-or-less Popperian, experimental science, since these would contradict the rigor required of mathematics, internally. I believe the solution is to conceive of paramathematics – and this appears to be happening in the first instances of the new sensibility in existence –, on Bruner’s second mode, the *narrative*. The goal of paramathematics must be to construct complex, causality-driven, linear discourses, that enrich mathematics by understanding it in context, historical, philosophical, cognitive, utilitarian, intra-scientific, aesthetic – or combinations of the above.

Elements of this kind of discourse have been apparent in the last two decades, coming from a variety of sources, by authors who are either mathematicians or, if I may say it, paramathematicians. Apart from the non-fictional and fictional narratives already mentioned – I add to these the novel *Turing's Smile* by Berkeley professor Christos Papadimitriou, to be published soon by MIT Press –, I mention without hope of exhausting this new treasure grove: the expository-historical work of Philip J. Davis, Ian Stewart, Keith Devlin; the brave, invigorating 'opinions', on his website, of Doron Zeilberger; the work of philosopher-historian of mathematics Leo Corry; the books and columns of John Allen Paulos; the wise books of Reuben Hersh, especially *What is Mathematics, Really?;* the interviews of Alain Connes; the interviews and provocative essays of Gregory Chaitin; the brilliant narrative book *The Universal Computer* by Martin Davis, which traces the creation of the computer to the crisis in the foundations of mathematics, an inspiration for how profound history of mathematics can become. And, before these, the pioneering book *Mathematics, the loss of certainty*, by Morris Kline and Douglas Hofstadter's fascinating *Gödel, Escher, Bach*.

What characterizes the works listed, is a passion to talk about mathematics in a new way. What unites them is the indifference to the puritanical, "pure" platonic mindset, by which a specialist mathematician – and all mathematicians are specialists –, may speak either absolutely rigorously about the narrow field of his knowledge or else shut up. That he must always prove everything he says, that he must always be totally precise, totally clear, totally understandable – to the 4 or 5 people in the world that understand him anyway.

When mathematicians come to the real world, it is usually to ask for money – in this they resemble a lot the monastic communities of the Holy Mountain. Paramathematics can also amend this rather embarrassing situation, providing an ideal way-station of ideas, this way and that, where the great work done in mathematics can find a form that is understandable and relevant to our world.

Of course, we shall always need the theorem provers, the street-fighters and the mountain-climbers – there can be no mathematics without them. But we shall also need mathematical thinking that is at least once-removed from formalist rigor. And narrative can provide a solid ground for it.

But as hard-core mathematicians of the old variety – like Uncle Petros, eh? – are against frivolous novelties and think that a new theory or field is justified only if it can throw new light on difficult problems, let me just list a few:

1. Is the proof of the Four Color Theorem, by Appel and Haken, with its heavy reliance on computing a real mathematical proof?
2. To what extent did Andrew Wiles really prove Fermat's Last Theorem. I am not of course – am not in a position to, anyway – doubting the validity of the proof per se. But is it valid to ask, how many thousands or millions of deductive steps from the original statement of a problem can the labyrinthine logic of deduction validate itself? In theory, of course, this can happen ad infinitum. But does such certainty have place in the grand scale, after Gödel – and a less Platonist view of mathematics. I also mention the doubts expressed by Doron Zeilberger: that the proof stands philosophically on extremely dubious ground, straining use of the axiom of choice. Remember the consistency of set theory has not yet been proven. It can be undecidable or inconsistent. What happens if it is proven not to be so? Also, the phrase that it is not 'psychologically satisfying' points to interesting directions of investigation – directions that have to be paramathematical, I believe, rather than strictly formal-mathematical.
3. What is the state of the truths expressed by Gödel's great theorems in the light of algorithmic information theory? Leading to:
4. Is there meaning in the recent talk of an "experimental mathematics", an approach not based on the Euclidean-Hilbertian criterion of well-formedness, but on a model closer to the Popperian?

These are some of the questions that are grand and crucial to mathematics. Yet mathematics in itself seems unable to take them further. Outside help is needed: paramathematics.

I shall now conclude by trying to put my ideas into a four-point a proposal – to serve as a working hypothesis.

Point one

Mathematical narrative must enter the school curriculum, in both primary and secondary education. The aim is: a) to increase the appeal of the subject, b) to give it a sense of intellectual, historical and social relevance and a place in our culture, c) to give students a better sense of the scope of the field, beyond the necessarily limited technical mathematics that can be taught within the constraints of the school system.

Point two

Mathematical narrative must supplement and interact with technical mathematics teaching. But a substantial amount of time now given to technical mathematics – the only kind that is taught –, should, according to the age and the level of the students, be taken over by narrative mathematics. It is a certain fact that at least 90% of the students will forget at least 90% of the technical mathematics they are taught anyway, before the end of their education. So, use some of the time to make sure what they are taught sticks. Save time for narrative, use it to embed mathematics in the soul.

Point three

The early years of schooling are crucial, as it is often here that the dislike of mathematics is planted. The main cause of this is the difficulty of a young child accepting abstraction and irrelevance – which mostly peaks with the introduction of the concept of number and arithmetic operations. At age five or six, a child lives in a storied internal environment, i.e. an environment cognitively organized by stories of all kinds, of family, of home, of daytime routine, of behavior, of neighborhood, of games, of friends, of animals, of dream. The main characteristics of the storied world are integration and emotional richness. With the introduction to mathematics, the child is *de-storied*, a neologism that sounds suspiciously close to “destroyed”. We must be very careful when we provide the first bites of the fruit of the Tree of Abstract Knowledge.

The first way to use stories at this level is to create storied environments for mathematics, global as well as local, to ease the transition into abstraction. To create stories, possibly with identifiable heroes and structured but rich conceptual environments, amenable to accepting independently fitting sub-stories. In this context heroes (targets for identification by the child) will progress doing mathematics, the students assisting him by participating. The mathematics itself need not be, at first numerical – no, nor Bourbakiste. One of the basic tasks of mathematical education is to teach people how to think. Yet, while it is highly debatable that solving second degree equations or complex trigonometric identities will promote thinking, it is obvious that puzzles, mysteries, whodunits, riddles, codes, anagrams, geometrical challenges – structures that can interact with stories by their very nature –, into which the concepts of number and operations are gradually introduced, are directly related to the basic logical operations. Using Euclid's Poetics, these can be either parts of larger narratives or mini-stories in themselves.

A storied environment is ideal to assimilate these, as basic variations of the Quest Myth – an exciting instance here being variations on Lewis Caroll's *Alice in Wonderland*. Furthermore, by use of Euclid's Poetics, there can be a gradual transformation of story and quest problems into abstract ones, as the student progresses – and these without losing the context. The emotionally motivated storied quest should allow teamwork – don't forget, history of mathematics teaches us that interaction and collaboration play a huge part in discovery –, providing an alternative to the solipsistic view of mathematical practice that a purist-Platonist mentality requires.

So, for our young students we can make rich stories, rich both in drama and in problem-solving strategies and tasks – these, we saw, can become interchangeable using Euclid's Poetics. Use what storytellers know about good stories and mathematicians about good proofs – Polya's work here could provide great guidelines – to best advantage.

Point four

In the existing situation, as students grow and acquire basic skills, the mathematics-to-mathematics-education homomorphism usually takes the following form: by Ërdös dictum that 'mathematicians are machines turning coffee into theorems', we strive in our schools to make students 'machines turning milder or stronger forms of coercion into exercise-solving'.

The exclusive emphasis on the technical side of mathematics (mathematics reduced to mechanical routines that even many mathematicians dislike) has the ridiculous effect that while a fourteen-year-old is taught in the literature class Homer (in a Greek school) or Shakespeare and Dickens (in an English), i.e. the crowns of achievement in the field, the place within the grand scheme of mathematics of the math they are taught at the same time is, by analogy *Mary had a little lamb* or *Jingle Bells*. Of course, we cannot teach high school students advanced algebraic topology or Selberg's 'elementary' proof of the Prime Number Theorem or suchlike stuff. But we can talk *about them* – and if we think we cannot that's only because in mathematics, we have insanely equated teaching something – thanks uncle Plato! thanks uncle Euclid! – with a rigorous mastery of technical tricks, a mastery which we know deep down is, without Husserl's willful intention to learn, nothing but the application of a bundle of recipes. Oh, short term memory is a wondrous thing, as all of us know who have excelled in an exam by studying intensely the last two days – and then forgetting everything a week later. But in this case, rigor without emotion is about as useful as rigor mortis.

So: the answer is to introduce mathematical biography and history in the syllabus. And here we see again the importance of paramathematics. For otherwise, if narrative has its role in mathematics itself, its use in mathematics education (by a negative use of the homomorphism, math to math-ed) can be considered illegitimate or, if not, second-rate.

Through the teaching of mathematical biography and history with a paramathematical slant, students can a) identify with the human context of mathematical research and thus be better motivated for learning, b) depending on the mathematician or the period or problem that is taught, a lot of the relevant technical mathematical material can be integrated, in ways which, using Euclid's Poetics, can smoothen and motivate the transitions, from person to idea to problem to person, c) provide and help students find a context for mathematics and through it a sense of real purpose in what is to them now a mostly meaningless playing around with abstract symbols. The fact that some students like it, does not mean it has meaning for them: people like chess or solitaire, though they are meaningless.

By integrating biography with the history of mathematics, from a paramathematical viewpoint, as a form of fascinating intellectual history of ideas, the technical knowledge taught to students can acquire a true context. All good stories, we said, are problem-generated and goal-oriented, as is mathematics. Let us endow, through stories, our teaching of mathematics with both the problems that generated it and the goals to which it's oriented.

It is degrading to both the students and mathematics itself, to present the reason for having to learn mathematics as, a) its necessity for some more advanced subject yet unmet and totally uncared-for, b) its – totally invisible – use in creating the rocket, the computer and the electronic toothbrush, c) its usefulness in 'learning to think' when this causal relationship is anything but apparent to the student – and not always to us. Let us, for the first time, try to answer question of the need for mathematics in the complex, meaningful way that it deserves.

Finally, I cannot but add, as an afterthought, an extra-mathematical extra point: we must try, at last, to make space for Howard Gardner's theory of Multiple Intelligences in our school system. For in mathematics, it is especially useful: for the students who have the aptitude, the teaching and the learning of mathematics can be significantly improved by the use of narrative. For some, those possessing for example a purely a verbal or physical inclination in their intelligence, by Gardner's classification, teaching mathematics in a technical way verges on the sadistic. Let's develop the sensitivity to view this, too, realistically. Some horses don't want to drink, no matter what we do. Some horses are just not as thirsty as others.

Ladies and Gentlemen,

What I have outlined, you may object, is not so much a plan as a dream.

Yet, the visible, measurable difference – however slight – that the mathematical narratives in existence have made in the last few years, the resurgence of interest in mathematics in the general culture and, above all, its investment with meaning and emotion through the appearance of paramathematics, give me hope that this is not only a correct, but a realistic direction.

Of course, if it is to come to fruition, both you, the specialists and researchers in mathematics education and we, the storytellers, have a daunting task. Even an experimental application of such a program, needs planning, hard work, both in the way of concrete research and in the preparation of the teaching material, in whatever form. The work has to be creative and inspired. Yet, the paramathematical sensibility already developing, makes me confident that both the people and the vision is there. And it is an added, enormous stimulus, that in being aligned to such a project – by the previous argument for paramathematics – we shall not be working merely to advance mathematics education, but mathematics itself. This is not a case of going down to the level of the students, but changing the paradigm, to provide a worthy context for a great and glorious subject – and by this change all will profit, even the subject itself.

I ask you:

Which are the best teachers? Those that love their students.

Which are the best students? Those that love their subject.

So, you should work on the love – this is where the problem lies. Embed mathematics in the soul by embedding it in history, by embedding it in story. By showing how it is lovely and adventurous – the stuff of the best quest myths. By showing how it was created by complex, adventurous, brave, struggling human beings. If you cannot teach or even show much of its beauty directly, for technical reasons, show it by showing the light it reflected on the faces of its discoverers.

If our rationale for teaching a subject is circular – “you must learn it because it is useful, because it has uses, because it is useful, because you will need it later, because it is useful” – we won’t go a long way. A developing human being is many things, and chief among them a poet, an adventurer and a problem-solver. Give the poetry, the adventure and the problems, through stories, both small stories of environment and large stories of culture. Grip the heart – and the brain will follow.

As for the mathematicians themselves: don’t expect too much help. Most of them are too far removed in their ivory towers to take up such a challenge. And anyway, they are not competent. After all, they are just mathematicians – what we need is paramathematicians, like you.... It is you who can be the welding force, between mathematics and stories, in order to achieve the synthesis.

It is you: your field, your discipline, by its very position, can bring people and mathematics together, by the means of stories. It will be a hard struggle, because to triumph you will have to fight the powers that be. So, seek first of all to inspire yourselves, to gather vision and strength. Through stories. And we, the pariahs, the imported labor, the storytellers, will help you, we will join in your cause – if you make it yours. You have our phone number. But you must make that call. Thank you!